VANDERBILT UNIVERSITY

MATH 4110 – PARTIAL DIFFERENTIAL EQUATIONS

HW 5 – Solutions

The problems below deal with solving PDEs via transformation methods. As we saw in class, many times these methods involve solving integrals that require advanced techniques. If you come across one of such integrals, you can quote the answer from a textbook, an online table of integrals, or a symbolic software like Mathematica or Maple. If you cannot find the answer in any of these sources, leave the integral indicated.

Question 1. Consider the following initial-value problem:

$$u_{tt} + 2du_t - u_{xx} = 0 \quad \text{in } \mathbb{R} \times (0, \infty), \tag{1a}$$
$$u = g \quad \text{on } \mathbb{R} \times \{t = 0\}, \tag{1b}$$
$$u_t = h \quad \text{on } \mathbb{R} \times \{t = 0\}, \tag{1c}$$

where $d > 0$ is a constant. Equation (1) is known as the telegraph equation.

(a) Applying the Fourier transform in the spatial variable only, show that $\hat{u}$ solves the following problem:

$$\hat{u}_{tt} + 2d\hat{u}_t + |y|^2\hat{u} = 0 \quad \text{in } \mathbb{R} \times (0, \infty), \tag{2a}$$
$$\hat{u} = \hat{g} \quad \text{on } \mathbb{R} \times \{t = 0\}, \tag{2b}$$
$$\hat{u}_t = \hat{h} \quad \text{on } \mathbb{R} \times \{t = 0\}. \tag{2c}$$

(b) Problem (2) is an ODE for $\hat{u}$ for each fixed y. Solve it by trying a solution of the form $\hat{u} = \beta e^{t\gamma}$, where β and γ can depend on y. You should find

$$\hat{u}(y, t) = \begin{cases} e^{-dt}(\beta_1(y)e^{\gamma(y)t} + \beta_2(y)e^{-\gamma(y)t}) & \text{if } |y| \leq d, \\ e^{-dt}(\beta_1(y)e^{i\delta(y)t} + \beta_2(y)e^{-i\delta(y)t}) & \text{if } |y| \geq d, \end{cases}$$

where $\gamma(y) = \sqrt{d^2 - |y|^2}$ with $|y| \leq d$, $\delta(y) = \sqrt{|y|^2 - d^2}$ with $|y| \geq d$, and β_1 and β_2 are selected such that

$$\hat{g}(y) = \beta_1(y) + \beta_2(y),$$

and

$$\hat{h}(y) = \begin{cases} \beta_1(y)(\gamma(y) - d) + \beta_2(y)(-\gamma(y) - d) & \text{if } |y| \leq d, \\ \beta_1(y)(i\delta(y) - d) + \beta_2(y)(-i\delta - d) & \text{if } |y| \geq d. \end{cases}$$

(c) Using the above, conclude that the solution is given by

$$\begin{aligned} u(x, t) = {} & \frac{e^{-dt}}{\sqrt{2\pi}} \int_{|y| \leq d} (\beta_1(y)e^{ixy + \gamma(y)t} + \beta_2(y)e^{ixy - \gamma(y)t})\, dy \\ & + \frac{e^{-dt}}{\sqrt{2\pi}} \int_{|y| \geq d} (\beta_1(y)e^{i(xy + \delta(y)t)} + \beta_2(y)e^{i(xy - \delta(y)t)})\, dy. \end{aligned}$$

(d) What happens when $t \to \infty$? Interpret your result and explain the meaning of the constant d.

Solution 1. (a) This follows immediately from the property

$$\left(\frac{\partial u}{\partial x}\right)^{\hat{}} = iy\hat{u}.$$

(b) Plugging $\hat{u} = \beta e^{t\gamma}$ into the equation, we find $\gamma^2 + 2d\gamma + |y|^2 = 0$, hence $\gamma = -d \pm \sqrt{d^2 - |y|^2}$. Therefore, the solution is as stated.

(c) This follows from the formula for the inverse Fourier transform.

(d) Because of the term e^{-dt}, the solution approaches zero as $t \to \infty$. This means that the corresponding "waves" are dissipating energy, as it could be guessed from comparing the term $+2du_t$ with the dissipation term for the harmonic oscillator studied in ODE courses.

Question 2. Consider the following initial-value problem:

$$iu_t + \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty), \tag{3a}$$

$$u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}, \tag{3b}$$

where i is the complex number $i^2 = -1$. Using the Fourier transform, show that the solution to (3) is

$$u(x,t) = \frac{1}{(4\pi i t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{\frac{i|x-y|^2}{4t}} g(y)\, dy.$$

Hint: follow what we did in class for the heat equation.

Remark: Equation (3) is nothing else but the Schrödinger equation with the potential equal to zero and $\frac{\hbar}{2\mu} = 1$. This last condition can always be achieved by a suitable choice of units.

Solution 2. Follow what we did in class for the heat equation, replacing t by it.

Question 3. Consider the following initial-value problem for the wave equation:

$$u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty), \tag{4a}$$

$$u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\}, \tag{4b}$$

$$u_t = 0 \quad \text{on } \mathbb{R}^n \times \{t = 0\}. \tag{4c}$$

Using the Fourier transform, show that the solution to (4) is

$$u(x,t) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{\hat{g}(y)}{2} (e^{i(x\cdot y + t|y|)} + e^{i(x\cdot y - t|y|)})\, dy,$$

where $i^2 = -1$. (Notice that the solution is real despite the presence of i.)
Hint: some ideas of problem (1) can be useful here.

Solution 3. It is pedagogical to consider the case when $u_t = h$ on $\mathbb{R}^n \times \{t = 0\}$, with h not necessarily zero. Taking the Fourier transform of (4) in such a situation gives

$$\hat{u}_{tt} + |k|^2 \hat{u} = 0 \quad \text{for } t > 0,$$

$$\hat{u} = \hat{g} \quad \text{for } t = 0,$$

$$\hat{u}_t = \hat{h} \quad \text{for } t = 0.$$

This is a second order ODE for $\hat{u}$, whose solution is

$$\hat{u}(k,t) = \hat{g}(k)\cos(t|k|) + \frac{\hat{h}(k)}{|k|}\sin(t|k|),$$

so that

$$u(x,t) = \left(\hat{g}(k)\cos(t|k|) + \frac{\hat{h}(k)}{|k|}\sin(t|k|) \right)^{\vee}.$$

Now setting $h = 0$ and using the explicit formula for the inverse Fourier transform as an integral, we obtain the result.