VANDERBILT UNIVERSITY

MATH 4110 – PARTIAL DIFFERENTIAL EQUATIONS

HW 4 - Solutions

Recall that in class we solved the the wave equation with Dirichlet boundary conditions by supposing that $u(x,t) = X(x)T(t)$. This is called the method of separation of variables. Below, you are asked to employ this method for solving other problems.

Question 1. Use separation of variables to solve the following initial-boundary value problem for the wave equation (the only difference from what was done in class is the boundary condition):

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 && \text{in } (0,L)\times(0,\infty), \\ u(x,0) &= f(x) && 0 \le x \le L, \\ u_t(x,0) &= g(x) && 0 \le x \le L, \\ u_x(0,t) &= 0 && t \ge 0, \\ u_x(L,t) &= 0 && t \ge 0. \end{aligned}$$

Solution 1. The separation of variables is done as in class. The difference is that now for the equation

$$X'' + \lambda X = 0,$$

we use the boundary conditions

$$X'(0) = X'(L) = 0.$$

We then find $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 0, 1, 2, \dots$ and

$$X_n(x) = \cos\left(\frac{n\pi x}{L}\right),\ n = 0, 1, 2, \dots .$$

Differently than what was done in class, here $n = 0$ is included. Using λ_n in the T equation now produces

$$\begin{aligned} T_0(t) &= a_0 + b_0 t, \\ T_n(t) &= a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right),\ n = 1, 2, 3, \dots \end{aligned}$$

The solution T_0 corresponds to using λ_0 in the equation for T. We obtain, after redefining the coefficients a_0 and b_0,

$$u(x,t) = \frac{a_0 + b_0 t}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right) \right) \cos\left(\frac{n\pi x}{L}\right),$$

where a_n and b_n are given by the familiar formulas for Fourier coefficients on $[0, L]$.

Question 2. Show that the solution you found in problem 1 can be written as a superposition of a forward and a backward wave.

Solution 2. We will write u as $u = u_1 + u_2 + \frac{a_0}{2}$, where u_1 is a forward wave, u_2 a backward wave, and $\frac{a_0}{2}$, being a constant, can be thought of as either a forward or backward wave.

Using the trigonometric identities

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$$

and

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

we find

$$u_1(x,t) = -\frac{b_0}{4c}(x-ct) + \sum_{n=1}^{\infty}\left(\frac{a_n}{2}\cos\left(\frac{n\pi(x-ct)}{L}\right) - \frac{b_n}{2}\sin\left(\frac{n\pi(x-ct)}{L}\right)\right),$$

and

$$u_2(x,t) = \frac{b_0}{4c}(x+ct) + \sum_{n=1}^{\infty}\left(\frac{a_n}{2}\cos\left(\frac{n\pi(x+ct)}{L}\right) + \frac{b_n}{2}\sin\left(\frac{n\pi(x+ct)}{L}\right)\right).$$

Question 3. Solve problem 1 with $c = 1$, $L = \pi$, $f(x) = \sin^3 x$, and $g(x) = \sin(2x)$.

Solution 3. Using the formulas for a_n and b_n we find

$$a_n = \frac{12(1+\cos(n\pi))}{\pi(9-10n^2+n^4)} \text{ for } n \geq 0,\ b_0 = 0, \text{ and } b_n = \frac{4(-1+\cos(n\pi))}{n(n^2-4)\pi} \text{ for } n \geq 1.$$

These formulas are well defined because the values of n that vanish the denominators ($n = 1, 3$ for a_n and $n = 2$ for b_n) correspond to vanishing coefficients.

Question 4. Use separation of variables to solve the following initial-boundary value problem for the heat equation:

$$\begin{aligned} u_t - ku_{xx} &= 0 && \text{in } (0,L)\times(0,\infty),\\ u(x,0) &= f(x) && 0 \leq x \leq L,\\ u(0,t) &= 0 && t \geq 0,\\ u(L,t) &= 0 && t \geq 0. \end{aligned}$$

Interpret your result.

Solution 4. Separating variables $u(x,t) = X(x)T(t)$ we find

$$X'' + \lambda X = 0,$$

and

$$T' + \lambda k T = 0.$$

Using the boundary conditions $X(0) = X(L) = 0$ we find $\lambda_n = (n\pi/L)^2$ and

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right),\ n = 1,2,3,\ldots$$

Using λ_n in the equation for T gives

$$T_n(t) = e^{-\frac{n^2\pi^2}{L^2}kt},\ n = 1,2,3,\ldots.$$

Hence

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2}{L^2}kt},$$

where

$$b_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx.$$

Since $e^{-\frac{n^2\pi^2}{L^2}kt} \to 0$ as $t \to \infty$, we see that $u \to 0$ as $t \to \infty$. This means that the temperature will eventually reach zero, as it should be for an insulated rod kept at zero temperature at its endpoints.

Question 5. Solve problem 4 with $k = 17$, $L = \pi$, and

$$f(x) = \begin{cases} 0, & 0 \le x \le \frac{\pi}{2}, \\ 2, & \frac{\pi}{2} < x \le \pi. \end{cases}$$

Discuss the convergence of the solution you found.

Solution 5. Computing

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx \\ &= \frac{4}{\pi n}\left(\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right). \end{aligned}$$

Thus

$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right)\sin(nx)\,e^{-17n^2t}.$$

Since

$$\left|\frac{1}{n}\left(\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right)e^{-17n^2t}\right| \le \frac{2}{n}e^{-17n^2t}$$

and the series

$$\sum_{n=1}^{\infty}\frac{2}{n}e^{-17n^2t}$$

converges for each $t > 0$, we conclude that the series for u does converge for each $t > 0$.