VANDERBILT UNIVERSITY

MATH 4110 – PARTIAL DIFFERENTIAL EQUATIONS

HW 1 Solutions

Question 1. Review multivariable calculus, especially the chain rule in several variables.

Solution 1. Done!

Question 2. Verify whether the given function is a solution of the given PDE:

(a) $u(x,y) = y\cos x + \sin y \sin x$, $u_{xx} + u = 0$.

(b) $u(x,y) = \cos x \sin y$, $(u_{xx})^2 + (u_{yy})^2 = 0$.

Solution 2. (a) Compute $u_{xx}(x,y) = -y\cos x - \sin x \sin y = -u(x,y)$, thus u is a solution.

(b) Compute $u_{xx}(x,y) = -\cos x \sin y$, $u_{yy}(x,y) = -\cos x \sin y$, thus $(u_{xx}(x,y))^2 + (u_{yy}(x,y))^2 = 2\cos^2 x \sin^2 y \neq 0$, hence u is not a solution.

Question 3. For each PDE below, identify the unknown function and state the independent variables. State the order of the PDE. Write the PDE in the form $F(x, u, Du, \dots, D^m u) = 0$, i.e., identify the function F. State if the PDE is homogeneous or non-homogeneous, linear or non-linear.

(a) $u_{tt} - u_{xx} = f$.

(b) $u_y + uu_x = 0$.

(c) $\displaystyle\sum_{i,j,k=1}^{n} a^{ijk}\partial^3_{ijk} v + v = 0,$

where i, j, k range from 1 to 3.

(d) $u_{xx} + x^2 y^2 u_{yy} = (x+y)^2$.

(e) $u_{xy} + \cos(u) = \sin(xy)$.

Solution 3. (a) Unknown: u. Independent variables: x, t. Order: second. We have

$$F(p_1, \dots, p_9) = p_9 - p_6 - f(p_1, p_2).$$

The equation is linear and non-homogeneous.

(b) Unknown: u. Independent variables: x, y. Order: first. We have

$$F(p_1, \dots, p_5) = p_5 + p_3 p_4.$$

The equation is non-linear (because of the term uu_x) and homogeneous.

(c) It is instructive to consider a slightly more general case, with i, j, k ranging from 1 to n. Unknown: v. Independent variables: $x^1, \dots, x^n$. Order: third. We have

$$F(x_1, \dots, x_n, p, p_1, \dots, p_n, p_{11}, \dots, p_{nn}, \dots, p_{111}, \dots, p_{nnn}) = \sum_{i,j,k=1}^{n} a^{ijk} p_{ijk} + p.$$

The equation is linear and homogeneous.

(d) Unknown: u. Independent variables: x, y. Order: second. We have

$$F(p_1, \dots, p_9) = p_6 + p_1^2 p_2^2 p_9 - (p_1 + p_2)^2.$$

The equation is linear and non-homogeneous.

(e) Unknown: u. Independent variables: x, y. Order: second. We have

$$F(p_1, \dots, p_9) = p_7 + \cos p_3 - \sin(p_1 p_2).$$

The equation is non-linear (because of $\cos u$) and non-homogeneous.

Question 4. Consider a PDE $F(x, u, Du, \dots, D^m u) = 0$ and let P be the operator associated with it. Prove that the PDE is linear if and only if P is a linear operator.

Solution 4. Suppose the PDE is linear. Thus,

$$F_H(x, u, Du, \dots, D^m u) = \sum_{k=0}^{m} F_k(x, D^k u), \tag{1}$$

where each F_k is a sum of linear functions on derivatives of u of order k, i.e.,

$$F_k(x, D^k u) = \sum_{\ell=1}^{n^k} F_{k\ell}(x, u^{(\ell)}), \tag{2}$$

where each $u^{(\ell)}$ represents one of the n^k possible derivatives of u of order k. Let u and v be two functions for which $F(x, u, Du, \dots, D^m u)$ and $F(x, v, Dv, \dots, D^m v)$ are well-defined, but are otherwise arbitrary, and let a and b be two arbitrary constants. Then

$$F_k(x, aD^k u + bD^k v) = a \sum_{\ell=1}^{n^k} F_{k\ell}(x, u^{(\ell)}) + b \sum_{\ell=1}^{n^k} F_{k\ell}(x, v^{(\ell)})$$

by the linearity of $F_{k\ell}$. Hence

$$F_H(x, au + bv, aDu + bv, \dots, aD^m u + bD^m v) = aF_H(x, u, Du, \dots, D^m u) + bF_H(x, v, Dv, \dots, D^m v).$$

Since by definition $Pu = F_H(x, u, Du, \dots, D^m u)$, we conclude

$$P(au + bv) = aF_H(x, u, Du, \dots, D^m u) + bF_H(x, v, Dv, \dots, D^m v) = aPu + bPv,$$

as desired.

Reciprocally, suppose that P is a linear operator. Then it can be written as

$$\begin{aligned} Pu = & \sum_{i_1 i_2 \cdots i_m} a^{i_1 i_2 \cdots i_m} \partial^m_{i_1 i_2 \cdots i_m} u + \sum_{i_1 i_2 \cdots i_{m-1}} a^{i_1 i_2 \cdots i_{m-1}} \partial^{m-1}_{i_1 i_2 \cdots i_{m_1}} u \\ & + \sum_{i_1 i_2 \cdots i_{m-2}} a^{i_1 i_2 \cdots i_{m-2}} \partial^{m-1}_{i_1 i_2 \cdots i_{m_2}} u + \cdots + \sum_{i_1 i_2} a^{i_1 i_2} \partial^2_{i_1 i_2} u + \sum_i a^i \partial_i u + au. \end{aligned}$$

This implies that F_H has the decomposition (1) with each F_k satisfying (2).

Question 5. Consider Maxwell's equations:

$$\begin{aligned} \operatorname{div} E &= \frac{\varrho}{\varepsilon_0}, \\ \operatorname{div} B &= 0, \\ \frac{\partial B}{\partial t} + \operatorname{curl} E &= 0, \\ \frac{\partial E}{\partial t} - \frac{1}{\mu_0 \varepsilon_0} \operatorname{curl} B &= -\frac{1}{\varepsilon_0} J, \end{aligned}$$

where div is the divergence and curl is the curl, also written

$$\operatorname{div} f = \nabla \cdot f, \text{ and } \operatorname{curl} f = \nabla \times f.$$

Assume that ϱ and J vanish. Show that Maxwell's equations then imply that E and B satisfy the wave equation:

$$\frac{\partial^2 E}{\partial t^2} - \frac{1}{\varepsilon_0 \mu_0} \Delta E = 0,$$

and

$$\frac{\partial^2 B}{\partial t^2} - \frac{1}{\varepsilon_0 \mu_0} \Delta B = 0.$$

Interpret your result. Can you guess what the constant $\frac{1}{\varepsilon_0 \mu_0}$ must equal to?

Solution 5. Under the assumptions, the equations become

$$\operatorname{div} E = 0, \tag{3}$$

$$\operatorname{div} B = 0, \tag{4}$$

$$\frac{\partial B}{\partial t} + \operatorname{curl} E = 0, \tag{5}$$

$$\frac{\partial E}{\partial t} - \frac{1}{\mu_0 \varepsilon_0} \operatorname{curl} B = 0. \tag{6}$$

Take the curl of (5) and note that $\operatorname{curl} \frac{\partial}{\partial t} = \frac{\partial}{\partial t} \operatorname{curl}$ to get

$$\frac{\partial}{\partial t} \operatorname{curl} B + \operatorname{curl} \operatorname{curl} E = 0.$$

But $\operatorname{curl} B = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$ by (6), thus

$$\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} + \operatorname{curl} \operatorname{curl} E = 0.$$

Recalling the following identity from multivariable calculus

$$\operatorname{curl} \operatorname{curl} f = \nabla(\operatorname{div} f) - \Delta f,$$

and using (3), we obtain the wave equation for E. The wave equation for B is similarly obtained.

The interpretation is that the electric and magnetic fields propagate in vacuum as waves. From the discussion about the wave equation in class, we conclude that $\frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ is the speed of propagation of the electromagnetic waves, which, from physics, we know to be equal to the speed of light (in vacuum).