The method of characteristics: This is a method for solving 1^{st} order PDEs upon transforming the PDE into a system of ODEs.

Consider the equation

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u). \quad (*)$$

Ex:

$$x^2 u\, u_x + \cos xy\, e^{u} u_y = \frac{u^2}{1+x^2+y^2}$$

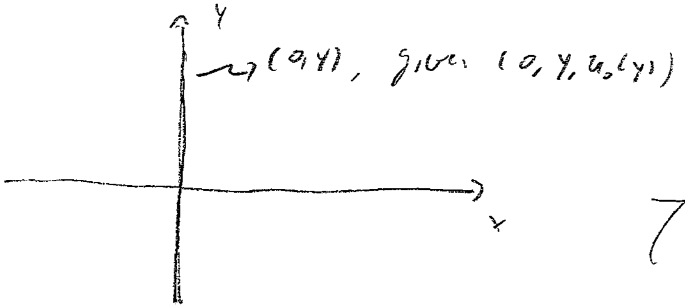
Equation $(*)$ is in general non-linear, but it is linear with respect to the derivatives of u, i.e.

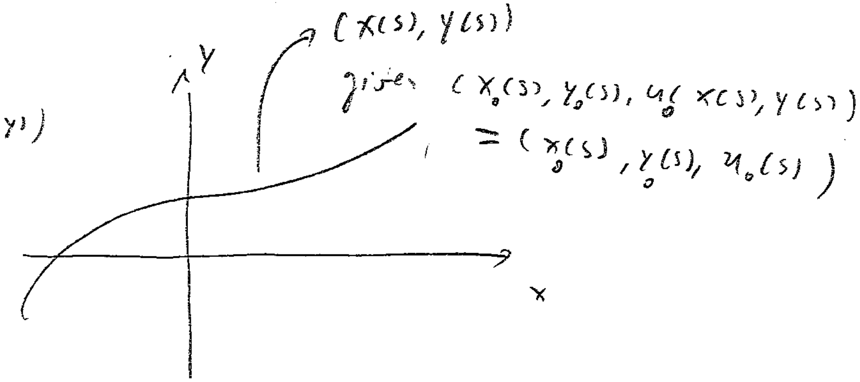
$$a(x,y,c_1u+c_2v)(c_1u+c_2v)_x = c_1 a(x,y,c_1u+c_2v)\,u_x + c_2\, a(x,y,c_1u+c_2v)\,v_x,$$

and similarly for b. Thus, we sometimes call $(*)$ a quasi-linear equation. When a, b and c do not depend on u, then $(*)$ is a linear equation.

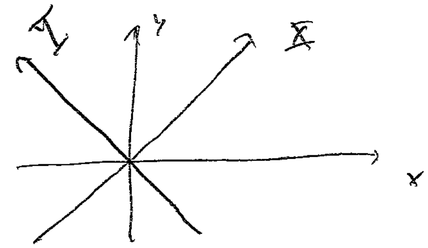
Consider first the linear case:

$a(x,y)\,u_x + b(x,y)\,u_y = c_0(x,y)\,u + f(x,y)$ (**). Assume we are given an initial condition for (**). The initial condition is typically of the form $u(0,y) = u_0(y)$, but it is convenient to consider that the initial condition is given along a curve on the xy-plane:





E.g, changing coordinates, $(0,y)$ becomes a curve $(\bar{X}(s), \bar{Y}(s))$:



So we write the initial condition as $\Gamma(s) = (x_0(s), y_0(s), u_0(s))$.
Write (**) as

$$(a, b, c_0 u + f)\cdot(u_x, u_y, -1) = 0$$

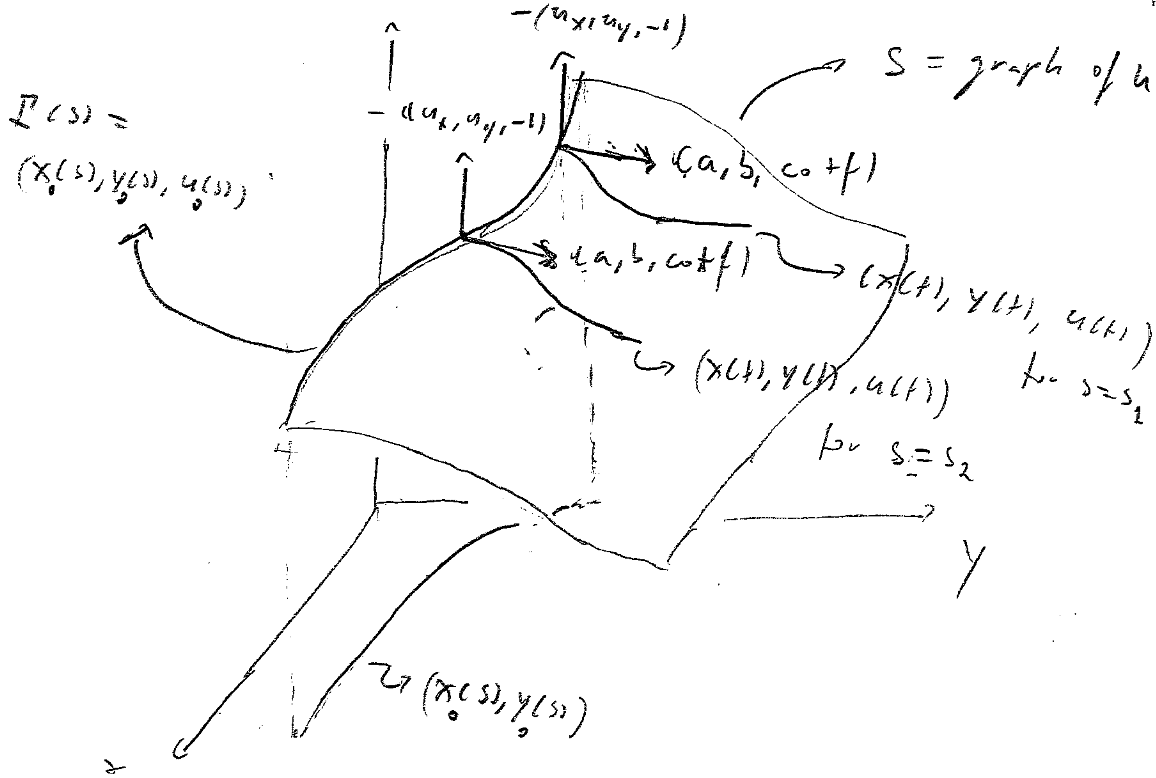
Recall (from multivariable calculus) that the graph of u can be viewed as a parametrized surface (or parametric surface) S: $r(x,y) = (x, y, u(x,y))$ and then $(u_x, u_y, -1)$ is normal to the surface S.

This means that $(a, b, cu+f)$ is tangent to S (for each (x,y))

Therefore, the equations:

$$\frac{dx}{dt}(t) = a(x(t), y(t)), \quad \frac{dy}{dt}(t) = b(x(t), y(t)), \quad \frac{du}{dt}(t) = c(x(t), y(t))\,u(t) + f(x(t), y(t)) \qquad (***)$$

defines curves $(x(t), y(t), u(t))$ that lie on S. The idea for solving the PDE is to solve $(***)$ with initial condition starting on $\Gamma(s)$



($(u_x, u_y, -1)$ points down, so we drew $-(u_x, u_y, -1)$). Notice that for each s we have a different curve, so we write

$x = x(t,s)$, $y = y(t,s)$

$u = u(t,s)$.

Notice that for each s parametrizing the initial conditions we have a different curve $(x(t), y(t), u(t))$, thus we write:

$$x = x(t,s), \quad y = y(t,s), \quad u = u(t,s)$$

Thus, we seek to solve

$$(G)\begin{cases} \dot{x}(t,s) = a(x(t,s), y(t,s)) \\ \dot{y}(t,s) = b(x(t,s), y(t,s)) \\ \dot{u}(t,s) = c(x(t,s), y(t,s))\, u(t,s) + f(x(t,s), y(t,s)) \end{cases}$$

with initial condition

$$x(0,s) = x_0(s), \quad y(0,s) = y_0(0,s), \quad u(0,s) = u_0(s),$$

where $\bullet$ means derivative with respect to t. Notice that the parameter s only enters in the initial condition, but we need to consider it when solving (G) (see ex. below)

The eq. (G) are called <u>characteristic equations</u> and its solutions <u>characteristic curves</u>