Non-homogeneous problem for the wave equation

Now we consider

$$\begin{cases} u_{tt} - \Delta u = f & \text{in } (0,\infty)\times\mathbb{R}^n \\ u = g,\ \partial_t u = h & \text{on } \{t=0\}\times\mathbb{R}^n \end{cases}$$

, where $f:(0,\infty)\times\mathbb{R}^n \to \mathbb{R}$ is given

By linearity, it suffices to consider the case when $g=h=0$.

We consider the family of problems parametrized by s for $u = u(t,x,s)$

$$\begin{cases} u_{tt}(\cdot,\cdot,s) - \Delta u(\cdot,\cdot,s) = 0 & \text{in } (s,\infty)\times\mathbb{R}^n \\ u(\cdot,\cdot,s) = 0,\quad u_t(\cdot,\cdot,s) = f(\cdot,s) & \text{on } \{t=s\}\times\mathbb{R}^n \end{cases}$$

which has already been solved. Set $u(t,x) = \int_0^t u(t,x,s)\,ds$, $x\in\mathbb{R}^n$, $t\geq 0$

<u>Thm</u> Let $f \in C^{[\frac{n}{2}]+1}([0,\infty)\times\mathbb{R}^n)$, $n\geq 2$. Define u as above. Then

(i) $u \in C^2([0,\infty)\times\mathbb{R}^n)$

(ii) $u_{tt} - \Delta u = f$ in $(0,\infty)\times\mathbb{R}^n$

(iii) $\displaystyle\lim_{\substack{(t,x)\to(0,x_0)\\ t>0}} u(t,x) = 0$, $\displaystyle\lim_{\substack{(t,x)\to(0,x_0)\\ t>0}} u_t(t,x) = 0$ for each $x_0 \in \mathbb{R}^n$.

Remark The idea of solving the inhomogeneous problem from the homogeneous one is known as "Duhamel's principle".

proof (i) $\left[\frac{n}{2}\right] + 1 = \frac{n+1}{2}$ if n is odd and $\left[\frac{n}{2}\right] + 1 = \frac{n+2}{2}$ if n is even. then u is C^2 by the previous theorems

(ii) Compute

$$\partial_t u(t,x) = \underbrace{u(t,x,s)\big|_{s=t}}_{=0} + \int_0^t u_t(t,x,s)\,ds = \int_0^t u_t(t,x,s)\,ds$$

$$\partial_t^2 u(t,x) = u_t(t,x,s)\big|_{s=t} + \int_0^t u_{tt}(t,x,s)\,ds = f(t,x) + \int_0^t \Delta u(t,x,s)$$

$$= f(t,x) + \Delta \int_0^t u(t,x,s) = f(t,x) + \Delta u(t,x).$$

(iii) Clear from the definition.

$\square$

The heat equation

We now investigate the Cauchy problem for the heat equation:

$$\begin{cases} u_t - \Delta u = 0 & \text{in } (0,\infty) \times \mathbb{R}^n \qquad (\text{usually } u_t - k\Delta u = 0,\ k > 0) \\ u = g & \text{on } \{t=0\} \times \mathbb{R}^n \end{cases}$$

Def. The function

$$\Phi(t,x) = \begin{cases} \dfrac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & , \quad x \in \mathbb{R}^n,\ t > 0 \\ 0 & , \quad x \in \mathbb{R}^n,\ t < 0 \end{cases}$$

is called the fundamental solution to the heat equation.

The motivation for introducing Φ is as follows. Since the heat equation involves two spatial derivatives and one time derivative, it has a natural scaling of the form $(t,x) \mapsto (\lambda^2 t, \lambda x)$. I.e., if u solves the heat equation, so does $u(\lambda^2 t, \lambda x)$. This suggests an Ansatz of the form $u(t,x) = \sigma\left(\frac{|x|^2}{t}\right)$, which leads to Φ.

Some computations

$$\partial_i \Phi(t,x) = -\frac{1}{(4\pi t)^{n/2}} \frac{1}{4t} \underbrace{\partial_i |x|^2}_{= \partial_i \sum x_j^2 = 2x_i} e^{-\frac{|x|^2}{4t}} = -\frac{1}{\pi^{n/2}} \frac{1}{(4t)^{\frac{n}{2}+1}} 2x_i \, e^{-\frac{|x|^2}{4t}}$$

$$\partial_i^2 \Phi(t,x) = \frac{1}{\pi^{n/2}} \left[\frac{4}{(4t)^{\frac{n}{2}+2}} x_i^2 - \frac{2}{(4t)^{\frac{n}{2}+1}} \right] e^{-\frac{|x|^2}{4t}}$$

$$\Delta \Phi(t,x) = \frac{1}{\pi^{n/2}} \frac{1}{(4t)^{\frac{n}{2}+1}} \left[\frac{4|x|^2}{4t} - 2n \right] e^{-\frac{|x|^2}{4t}}$$

$$\partial_t \Phi(t,x) = \frac{1}{(4\pi)^{n/2}} \left[-\frac{n}{2} \frac{1}{t^{\frac{n}{2}+1}} + \frac{1}{t^{\frac{n}{2}}} \partial_t \left(-\frac{|x|^2}{4t} \right) \right] e^{-\frac{|x|^2}{4t}} = \frac{1}{(4\pi)^{n/2}} \left[-\frac{n}{2t^{n/2+1}} + \frac{|x|^2}{4t^{\frac{n}{2}+2}} \right] e^{-\frac{|x|^2}{4t}}$$

$$= \frac{1}{\pi^{n/2}} \frac{1}{4^{n/2}} \left[-\frac{2n}{4t^{n/2+1}} + \frac{|x|^2}{4t^{n/2+1} \cdot t} \right] e^{-\frac{|x|^2}{4t}}$$

$$= \frac{1}{\pi^{n/2}} \frac{1}{(4t)^{n/2+1}} \left[-2n + \frac{|x|^2}{t} \right] e^{-\frac{|x|^2}{4t}}$$

Thus $\partial_t \Phi - \Delta \Phi = 0$

Next, notice that Φ and its derivatives approach zero as $t \to 0$ for $x \neq 0$. Hence we can think of Φ as solving the heat equation for all $(t,x) \neq (0,0)$, and as a consequence $\Phi(t, x-y)$ solves the equation for $x \neq y$ for each $x \neq y$ (these remarks can be made rigorous with the theory of distributions).

We shall also need:

$$\int_{\mathbb{R}^n} \Phi(t,x)\,dx = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}}\,dx \qquad z_i = \frac{x_i}{\sqrt{4t}}, \quad \mathrm{Jac}(x \mapsto z) = (4t)^{n/2}$$

$$= \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-|z|^2}\,dz = \frac{1}{\pi^{n/2}} \prod_{i=1}^{n} \underbrace{\int_{-\infty}^{+\infty} e^{-z_i^2}\,dz_i}_{=\sqrt{\pi}} = 1.$$

Theo Let $g \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and define u by

$$u(t,x) = \int_{\mathbb{R}^n} \Phi(t, x-y)\,g(y)\,dy = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy, \quad x \in \mathbb{R}^n, \ t > 0.$$

Then

(i) $u \in C^\infty((0,\infty) \times \mathbb{R}^n)$

(ii) $u_t - \Delta u = 0$ in $(0,\infty) \times \mathbb{R}^n$

(iii) $\displaystyle\lim_{\substack{(t,x) \to (0,x_0) \\ t>0}} u(t,x) = g(x_0)$ for each $x_0 \in \mathbb{R}^n$

proof $\frac{1}{t^{n/2}} e^{-\frac{|x|^2}{4t}}$ is smooth with uniformly bounded derivatives of all orders on $[\delta,\infty)\times\mathbb{R}^n$ for each fixed $\delta>0$. Next, let $z\in\mathbb{R}^n$ and let $\varepsilon>0$ be given. Choose $\delta>0$ such that $|y-z|<\delta \Rightarrow |g(y)-g(x)|<\varepsilon$. If $|x-z|<\frac{\delta}{2}$ then

using $\int\Phi=1$

$$|u(t,x)-g(z)| = \left|\int_{\mathbb{R}^n}\Phi(t,x-y)g(y)\,dy - \int_{\mathbb{R}^n}\Phi(t,x-y)g(z)\right|$$

$$\leq \int_{B_\delta(z)}\Phi(t,x-y)|g(y)-g(z)|\,dy + \int_{\mathbb{R}^n\setminus B_\delta(z)}\Phi(t,x-y)|g(y)-g(z)|\,dy = I_1+I_2.$$

$I_1 \leq \int_{B_\varepsilon(z)}\Phi(t,x-y)\,\varepsilon\,dy = \varepsilon$. For any $y\in\mathbb{R}^n\setminus B_\delta(z)$ we have that if $|x-z|<\frac{\delta}{2}$ then

$|y-z|\geq\delta$ for $y\in\mathbb{R}^n\setminus B_\delta(z)$

$$|y-z| \leq |y-x|+|x-z| \leq |y-x|+\frac{\delta}{2} \leq |y-x|+\frac{1}{2}|y-z|$$

Thus $|y-x| \geq \frac{1}{2}|y-z|$, so $e^{-\frac{|y-x|^2}{4t}} \leq e^{-\frac{|y-z|^2}{16t}}$

Now we estimate

$$I_2 \le 2\|g\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus B_\delta(z)} \Phi(t, x-y)\, dy \le \frac{cte}{t^{n/2}} \int_{\mathbb{R}^n \setminus B_\delta(z)} e^{-\frac{|x-y|^2}{4t}} dy \le \frac{cte}{t^{n/2}} \int_{\mathbb{R}^n \setminus B_\delta(z)} e^{-\frac{|y-z|^2}{16t}} dy$$

Let $\frac{y-z}{\sqrt{t}} = w$, $\text{Jac}(y \mapsto w) = t^{n/2}$, and $B_\delta(z) = y\left(B_{\delta/\sqrt{t}}(0)\right)$ since $|y-z| \le \delta \Leftrightarrow |w| \le \delta/\sqrt{t}$

$$I_2 \le cte \int_{\mathbb{R}^n \setminus B_{\delta/\sqrt{t}}(0)} e^{-\frac{|w|^2}{16}} dw .$$

Since $e^{-\frac{|w|^2}{16}} \to 0$ uniformly as $|w| \to \infty$, if t is sufficiently small $I_2 \le \varepsilon$.

Thus $|u(t,x) - g(z)| \le 2\varepsilon$ for $|x-z| \le \frac{\delta}{2}$ and t sufficiently small.

Finally

$$u_t(t,x) - \Delta u(t,x) = \int_{\mathbb{R}^n} \left(\Phi_t(t, x-y) - \Delta_x \Phi(t, x-y)\right) g(y) = 0 .$$

$\square$

Remarks

- If $g \geq 0$, $g \not\equiv 0$, then $u(t,x) = \int_{\mathbb{R}^n} \Phi(t, x-y)\, g(y) > 0$ for $t > 0$, $x \in \mathbb{R}^n$. Hence the heat equation has "infinite speed of propagation," in contrast to the wave equation

- Note that u is C^∞ even if g is only continuous. Thus the heat equation "smooths out" the data, in contrast, again, with the wave equation.

- It is possible to establish mean value formulas, a maximum principle, and Duhamel's principle for the heat equation.

- Note that while the regularity of the wave equation is on $[0, \infty)$, for the heat equation we have $(0, \infty)$ (mention reversibility).