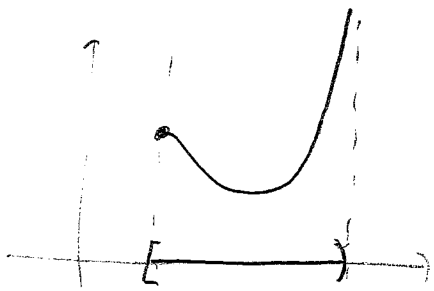
<u>Remark</u>: we will prove the theorem for $n \geq 3$, leaving the case $n = 2$ as exercise.

In order to prove the theorem, we will need the following facts.

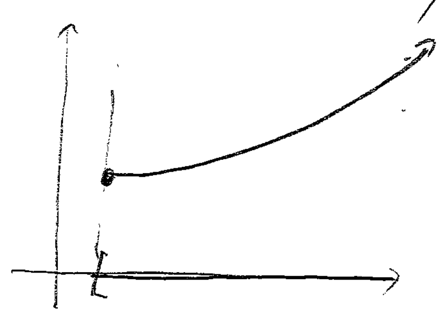
<u>Fact 1</u>: Let K be a closed and bounded set in $\mathbb{R}^n$. 'Bounded' means that K is contained in a ball of radius R and center zero for some R, i.e., $K \subseteq B_R(0)$. Let $f : K \to \mathbb{R}$ be continuous. Then f attains a maximum and a minimum within K, thus there exists constants a constant M such that $|f(x)| \leq M$ for all $x \in K$. The following pictures illustrate this fact:



K bounded but not closed

f continuous

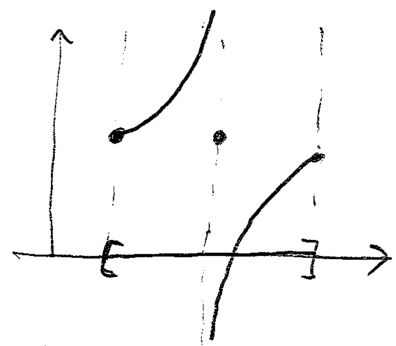
Fact 1 not true



K closed but not bounded

f continuous

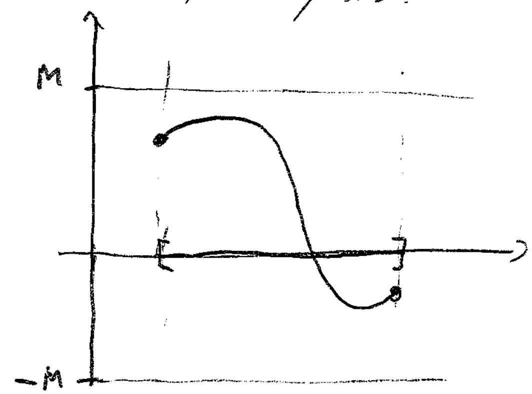
Fact 1 not true



K closed and bounded

f not continuous

Fact 1 not true



K closed and bounded

f continuous.

Similar pictures can be drawn in higher dimensions

Fact 2 Let $B_R(0)$ be the ball of radius R centered at the origin in $\mathbb{R}^n$. The volume element dx ($= dV$ in 3d calculus) in $B_r(0)$ can be written as

$dx = r^{n-1}\,dr\,d\Omega$, where $d\Omega$ is the volume element on $\partial B_1(0)$.

2d: $dx = r\,dr\,\underbrace{d\theta}_{d\Omega}$, 3d $dx = r^2 \sin\varphi\, dr\, d\varphi\, d\theta = r^2 dr \underbrace{\sin\varphi\, d\varphi\, d\theta}_{d\Omega}$

It follows that the volume element dS on $\partial B_r(0)$ is $dS = r^{n-1}\,d\Omega$

2d $dS = r\,d\theta$ 3d $dS = r^2 \sin\varphi\, d\varphi\, d\theta$

Fact 3 Let f be a C^2 function. Then

$$\underbrace{\frac{1}{\mathrm{vol}(\partial B_r(x_0))}\int_{\partial B_r(x_0)} f\, dS}_{\text{average of } f \text{ over } \partial B_r(x_0)} \longrightarrow f(x_0) \quad \text{as } r \to 0.$$

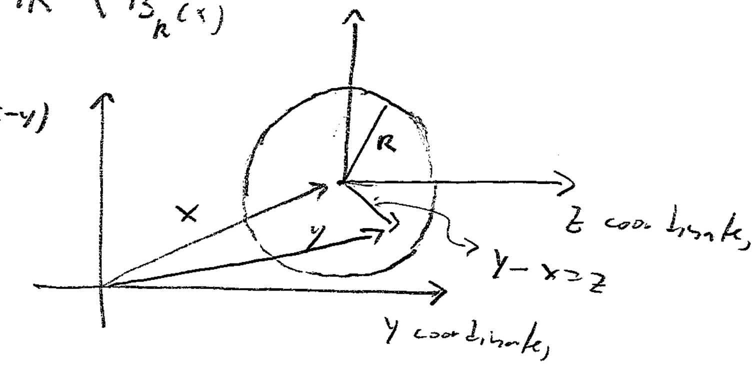
Note that $\mathrm{vol}(\partial B_r(x_0)) = n\alpha(n)\, r^{n-1}$

proof of the theorem:

To show that u is well-defined, fix $x \in \mathbb{R}^n$ and $R > 0$ (say, $R = 1$), and write

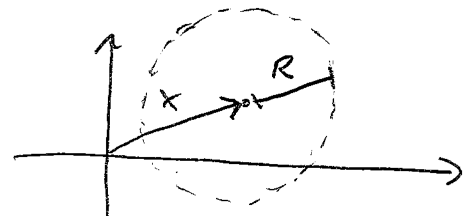
$$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y) f(y)\, dy = \int_{B_R(x)} \Gamma(x-y) f(y)\, dy + \int_{\mathbb{R}^n \setminus B_R(x)} \Gamma(x-y) f(y)\, dy$$

Change variables in the first integral: $z = y - x = -(x-y)$



$$\int_{B_R(x)} \Gamma(x-y) f(y)\, dy = \int_{B_R(0)} \Gamma(-z) f(z+x)\, dz \, ,$$

As z varies within $B_R(0)$, $z + x$ varies within $B_{R+|x|}(0)$.



Since $B_{R+|x|}(0) \subset \overline{B_{2(R+|x|)}(0)}$ and f attains a maximum and a minimum within $\overline{B_{2(R+|x|)}(0)}$, we have that $|f(z+x)| \le M$ for some M and all $z \in B_R(0)$.

Next, $$\Gamma(-z) = \frac{1}{n(n-2)\alpha(n)} \frac{1}{|-z|^{n-2}} = \frac{1}{n(n-2)\alpha(n)} \frac{1}{|z|^{n-2}}$$

Hence: $\quad |\Gamma(-z)| = \Gamma(-z)$ since $\Gamma \geq 0$

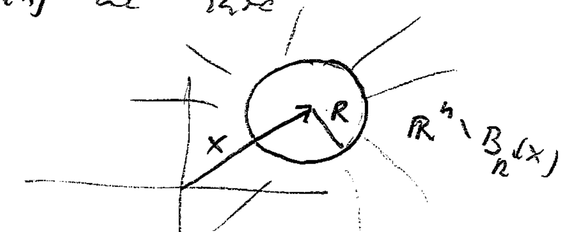
$$\left| \int_{B_R(0)} \Gamma(-z) f(z+x)\, dz \right| \leq \int_{B_R(0)} \widetilde{\Gamma}(z) |f(z+x)|\, dz \leq M \int_{B_R(0)} \Gamma(z)\, dz = \frac{M}{n(n-2)\alpha(n)} \int_{B_R(0)} \frac{1}{|z|^{n-2}}\, dz$$

Integrating in polar coordinates: $\displaystyle\int_{B_R(0)} \frac{1}{|z|^{n-2}}\, dz = \int_0^R \int_{\partial B_r(0)} \frac{1}{r^{n-2}}\, dS = \int_0^R \frac{1}{r^{n-2}} \int_{\partial B_1(0)} r^{n-1}\, d\Omega$

$\displaystyle = n\alpha(n) \int_0^R r\, dr = \frac{n\alpha(n)}{2} R^2$. Thus $\displaystyle\left| \int_{B_R(x)} \Gamma(x-y) f(y)\, dy \right| < \infty.$

For the second integral, note that for any $y \in \mathbb{R}^n \setminus B_R(x)$ we have

$$\Gamma(x-y) = \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x-y|^{n-2}} \leq \frac{1}{n(n-2)R^{n-2}} \quad \text{since } |x-y| \geq R$$



Since f has compact support and is continuous, we have $|f(y)| \leq M$ for all $y \in \mathbb{R}^n$,
thus $\displaystyle\left| \int_{\mathbb{R}^n \setminus B_R(x)} \Gamma(x-y) f(y)\, dy \right| \leq \int_{\mathbb{R}^n \setminus B_R(x)} \Gamma(x-y) |f(y)|\, dy < \infty$, hence $|u(x)| < \infty$, showing (i).

Remark: We could have used $|f(y)| \leq M$, $y \in \mathbb{R}^n$, in the estimate of the first integral, thus avoiding the argument with $B_{|x|+R}(0)$ etc. But we think it is useful to show that the first integral is $< \infty$ without using that f has compact support, as that trick is generally useful.

To show (ii), change variables:

$$u(x) = \int_{\mathbb{R}^n} \Gamma(\underbrace{x-y}_{z}) f(y)\, dy = \int_{\mathbb{R}^n} \Gamma(z) f(x-z) \overbrace{\left|\det\left|\frac{\partial z_i}{\partial y_j}\right|\right|}^{=1} dz = \int_{\mathbb{R}^n} \Gamma(y) f(x-y)\, dy$$

For $h \neq 0$ and $e_i = (0,0,\ldots,\underset{\substack{\downarrow \\ i\text{th component}}}{1},\ldots,0,0)$, we have

$$\frac{u(x+he_i) - u(x)}{h} = \int_{\mathbb{R}^n} \Gamma(y) \frac{f(x+he_i-y)}{h} dy - \int_{\mathbb{R}^n} \Gamma(y) \frac{f(x-y)}{h} dy = \int_{\mathbb{R}^n} \Gamma(y) \left(\frac{f(x+he_i-y) - f(x-y)}{h} \right) dy$$

But $\dfrac{f(x+he_i-y) - f(x-y)}{h} \to \dfrac{\partial f}{\partial x_i}(x-y)$ uniformly on $\mathbb{R}^n$ as $h \to 0$. Since $\dfrac{\partial f}{\partial x_i}$ is continuous, the same argument as in (i) shows that $\int_{\mathbb{R}^n} \Gamma(y) \dfrac{\partial f}{\partial x_i}(x-y)\, dy < \infty$, thus $\dfrac{\partial u}{\partial x_i}(x) = \lim_{h \to 0} \dfrac{u(x+he_i) - u(x)}{h}$ exists and

$$\frac{\partial u}{\partial x_i}(x) = \int_{\mathbb{R}^n} \Gamma(y) \frac{\partial f}{\partial x_i}(x-y)\, dy.$$

Using that $\dfrac{\partial f}{\partial x_i}$ is continuous (because f is C^2), we can then show that $\dfrac{\partial u}{\partial x_i}$ is continuous, thus $u \in C^1(\mathbb{R}^n)$.

To show that $\dfrac{\partial^2 u}{\partial x_i \partial x_j}$ exists, we repeat the above arguments applied to $\dfrac{\partial u}{\partial x_i}$.

To show that $\dfrac{\partial^2 u}{\partial x_i \partial x_j}$, we then use the continuity of $\dfrac{\partial^2 f}{\partial x_i \partial x_j}$. This establishes (i).

Exercise: complete the proof of (ii) by developing the above arguments.

We now move to establish (iii). Since we showed u to be C^2, we can apply Δ to it. We find

$$\Delta u(x) = \int_{\mathbb{R}^n} \Gamma(y) \Delta_x f(x-y)\, dy$$ where we use the notation Δ_x to indicate that the derivatives in Δ are with respect to x.

Let $\varepsilon > 0$ and consider $B_\varepsilon(0)$. We can split the integral as

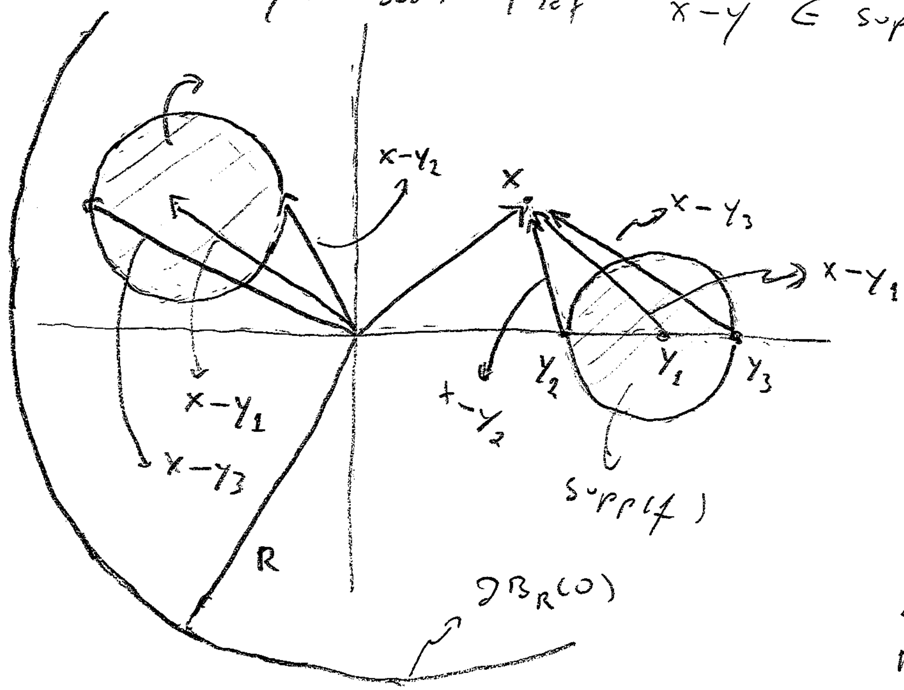
$$\Delta u(x) = \int_{\mathbb{R}^n} \Gamma(y) \Delta_x f(x-y)\, dy = \lim_{\varepsilon \to 0} \left[\int_{B_\varepsilon(0)} \Gamma(y) \Delta_x f(x-y)\, dy + \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Gamma(y) \Delta_x f(x-y)\, dy \right]$$

Since $\Delta_x f(x-y)$ is continuous and has compact support (because $f \in C_c^2(\mathbb{R}^n)$), we know that $|\Delta_x f(x-y)| \leq M$ for some constant M and all $x, y \in \mathbb{R}^n$. Thus

$$\left| \int_{B_\varepsilon(0)} \Gamma(y) \Delta_x f(x-y)\, dy \right| \leq M \int_{B_\varepsilon(0)} \frac{1}{n(n-2)\alpha(n)} \frac{1}{|y|^{n-2}}\, dy = \frac{M}{n(n-2)\alpha(n)} \int_0^\varepsilon \int_{\partial B_1(0)} d\Omega \, \frac{r^{n-1}}{r^{n-2}}\, dr$$

$= \text{constant} \cdot \varepsilon^2$. Thus $\displaystyle\lim_{\varepsilon \to 0} \int_{B_\varepsilon(0)} \Gamma(y) \Delta_x f(x-y)\, dy = 0.$

For the second integral, note that $f(x-y) \neq 0$ only when $x-y \in \text{supp}(f)$. Thus, when we integrate over y, $\int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Gamma(y)\,\Delta_x f(x-y)\,dy$ will not be necessarily zero only for those y's such that $x-y \in \text{supp}(f)$, or equivalently for $y \in x - \text{supp}(f)$



Therefore, we can choose R so large that $x - \text{supp}(f) \subseteq B_R(0)$ and $f(x-y)$ and its derivatives vanish on $\partial B_R(0)$. This is possible because f has compact support.

$$\int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Gamma(y)\,\Delta_x f(x-y)\,dy = \int_{B_R(0)\setminus B_\varepsilon(0)} \Gamma(y)\,\Delta_x f(x-y)\,dy$$

Remark: The picture is merely illustrative, supp(f) need not to be a ball, x and zero may belong to supp(f) or x - supp(f)

Next, notice that $\Delta_x f(x-y) = \Delta_y f(x-y)$.

To see this, denote

$$z = x-y$$

Compute $\displaystyle \frac{\partial}{\partial x_i}(f(z)) = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(z)\frac{\partial z_j}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(z) \overbrace{\frac{\partial}{\partial x_j}(x_i - y_i)}^{= \begin{cases} 0, & j \neq i \\ 1, & j = i \end{cases}} = \frac{\partial f}{\partial z_i}(z)$

Taking another derivative we find $\displaystyle \frac{\partial^2}{\partial x_i^2}(f(z)) = \frac{\partial^2 f}{\partial z_i^2}(z)$. Thus $\Delta_x(f(z)) = \Delta_z f(z)$.

On the other hand, $\displaystyle \frac{\partial}{\partial y_i}(f(z)) = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(z)\frac{\partial z_j}{\partial y_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(z) \underbrace{\frac{\partial}{\partial y_i}(x_j - y_j)}_{= \begin{cases} 0, & j \neq i \\ -1, & j = i \end{cases}} = -\frac{\partial f}{\partial z_i}(z)$

Taking another derivative, we get another negative, thus

$$\Delta_y(f(z)) = \Delta_z f(z) = \Delta_x(f(z)),$$ as claimed. Therefore,

we write $dS(y)$ to indicate that the variable being integrated on the boundary is y.

$$\int_{B_R(0)\setminus B_\varepsilon(0)} \Gamma(y)\,\Delta_x f(x-y)\,dy = \int_{B_R(0)\setminus B_\varepsilon(0)} \Gamma(y)\,\Delta_y f(x-y)\,dy \underset{\substack{\text{integrate} \\ \text{by parts}}}{=} -\int_{B_R(0)\setminus B_\varepsilon(0)} \nabla\Gamma(y)\cdot\nabla_y f(x-y)\,dy + \int_{\partial(B_R(0)\setminus B_\varepsilon(0))} \Gamma(y)\,\nabla_y f(x-y)\cdot\nu\,dS(y)$$

where we use ∇_y to denote the gradient on the y-variable. Since $\partial(B_R(0)\setminus B_\varepsilon(0)) = \partial B_R(0) \cup \partial B_\varepsilon(0)$ (see picture in the previous page) we have

$$\int_{\partial(B_R(0)\setminus B_\varepsilon(0))} \Gamma(y)\,\nabla_y f(x-y)\cdot\nu\,dS(y) = \int_{\partial B_R(0)} \Gamma(y)\,\underbrace{\nabla_y f(x-y)}_{=0 \text{ on } \partial B_R(0)}\cdot\nu\,dS(y) + \int_{\partial B_\varepsilon(0)} \Gamma(y)\,\nabla_y f(x-y)\cdot\nu\,dS(y) = \int_{\partial B_\varepsilon(0)} \Gamma(y)\,\nabla_y f(x-y)\cdot\nu\,dS(y)$$

Therefore

$$\int_{B_R(0)\setminus B_\varepsilon(0)} \Gamma(y)\,\Delta_y f(x-y)\,dy = -\overbrace{\int_{B_R(0)\setminus B_\varepsilon(0)} \nabla\Gamma(y)\cdot\nabla_y f(x-y)\,dy}^{\text{integrate by parts again}} + \int_{\partial B_\varepsilon(0)} \Gamma(y)\,\nabla_y f(x-y)\cdot\nu\,dS(y)$$

$$= -\left[-\int_{B_R(0)\setminus B_\varepsilon(0)} \nabla\cdot\nabla\Gamma(y)\,f(x-y)\,dy + \int_{\partial(B_R(0)\setminus B_\varepsilon(0))} \nabla\Gamma(y)\cdot\nu\,f(x-y)\,dS(y)\right] + \int_{\partial B_\varepsilon(0)} \Gamma(y)\,\nabla_y f(x-y)\cdot\nu\,dS(y).$$

But $\nabla\cdot\nabla\Gamma(y) = \Delta\Gamma(y) = 0$ for $y \neq 0$ (recall that Γ solves Laplace's equation except at the origin), thus the first integral vanishes. The second integral becomes an integral only over $\partial B_\varepsilon(0)$ by the same argument used above. Therefore:

$$\int_{\mathbb{R}^n\setminus B_\varepsilon(0)} \Gamma(y)\,\Delta_x f(x-y)\,dy = -\int_{\partial B_\varepsilon(0)} \nabla\Gamma(y)\cdot\nu\,f(x-y)\,dS(y) + \int_{\partial B_\varepsilon(0)} \Gamma(y)\,\nabla_y f(x-y)\cdot\nu\,dS(y).$$

We now want to take the limit $\varepsilon \to 0$. Look first at the second integral:

$$\left|\int_{\partial B_\varepsilon(0)} \Gamma(y)\,\nabla_y f(x-y)\cdot\nu\,dS(y)\right| \leq \int_{\partial B_\varepsilon(0)} |\Gamma(y)|\,|\nabla_y f(x-y)\cdot\nu|\,dS(y).$$

Recall the following property of the dot product: $|v\cdot u| \leq |v|\,|u|$.

(called Cauchy-Schwarz inequality)

Then $|\nabla_y f(x-y)\cdot \nu| \le |\nabla_y f(x-y)| \underbrace{|\nu|}_{=1} = |\nabla_y f(x-y)|$. Since $f \in C_c^2(\mathbb{R}^n)$, ∇f is continuous and has compact support, thus by fact 1 there exists a constant M such that $|\nabla f(z)| \le M$ for all $z \in \mathbb{R}^n$. Hence

$$\int_{\partial B_\varepsilon(0)} |\Gamma(y)|\, |\nabla_y f(x-y)\cdot \nu|\, dS(y) \le \frac{M}{n(n-2)\alpha(n)} \int_{\partial B_\varepsilon(0)} \frac{1}{|y|^{n-2}}\, dS(y) = \frac{M}{n(n-2)\alpha(n)} \int_{\partial B_\varepsilon(0)} \frac{1}{|y|^{n-2}}\, dS(y)$$

On $\partial B_\varepsilon(0)$, $|y| = \varepsilon$ and $dS(y) = \varepsilon^{n-1}\, d\Omega$, so we get

$$\frac{M}{n(n-2)\alpha(n)} \int_{\partial B_1(0)} \frac{1}{\varepsilon^{n-2}}\, \varepsilon^{n-1}\, d\Omega = \text{constant}\cdot \varepsilon \longrightarrow 0 \text{ as } \varepsilon \to 0, \text{ so}$$

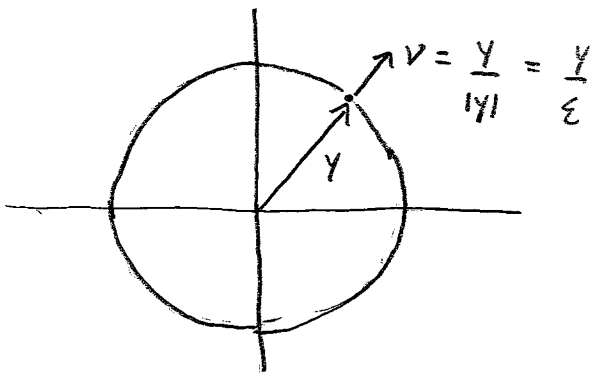
$$\lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon(0)} \Gamma(y)\, \nabla_y f(x-y)\cdot \nu\, dS(y) = 0.$$ It remains to analyze

$$-\int_{\partial B_\varepsilon(0)} \nabla \Gamma(y)\cdot \nu\, f(x-y)\, dS(y).$$

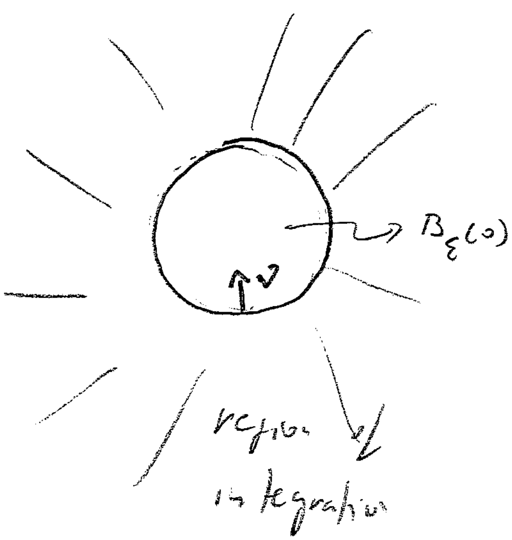
Recall that we computed $\dfrac{\partial \Gamma(y)}{\partial y_i} = -\dfrac{1}{n\alpha(n)} \dfrac{y_i}{|y|^n}$, therefore we have

$$\nabla \Gamma(y) = \left(\frac{\partial \Gamma(y)}{\partial y_1}, \ldots, \frac{\partial \Gamma(y)}{\partial y_n} \right) = \left(-\frac{1}{n\alpha(n)} \frac{y_1}{|y|^n}, \ldots, -\frac{1}{n\alpha(n)} \frac{y_n}{|y|^n} \right) = -\frac{1}{n\alpha(n)} \frac{1}{|y|^{n-1}} \left(\frac{y_1}{|y|}, \ldots, \frac{y_n}{|y|} \right)$$

Since on $\partial B_\varepsilon(0)$ $|y| = \varepsilon$, we can write $\nabla \Gamma(y) = -\dfrac{1}{n\alpha(n)} \dfrac{1}{\varepsilon^{n-1}} \left(\dfrac{y_1}{\varepsilon}, \ldots, \dfrac{y_n}{\varepsilon} \right)$.



Notice that $\left(\dfrac{y_1}{\varepsilon}, \ldots, \dfrac{y_n}{\varepsilon} \right) = \dfrac{y}{\varepsilon}$ is the unit outer normal to $\partial B_\varepsilon(0)$. However, in our case the unit outer normal points to the inside of the ball because the original domain of integration was exterior to $B_\varepsilon(0)$, thus $\dfrac{y}{\varepsilon} = -\nu$. Hence



$$\nabla \Gamma(y) = \frac{1}{n\alpha(n)} \frac{1}{\varepsilon^{n-1}} \nu. \quad \text{Therefore}$$

$$\nabla \Gamma(y) \cdot \nu = \frac{1}{n\alpha(n)} \frac{1}{\varepsilon^{n-1}} \nu \cdot \nu = \frac{1}{n\alpha(n)} \frac{1}{\varepsilon^{n-1}}$$

We have

$$-\int_{\partial B_\varepsilon(0)} \nabla \Gamma(y)\cdot \nu\, f(x-y)\, dS(y) = -\frac{1}{n\alpha(n)}\frac{1}{\varepsilon^{n-1}}\int_{\partial B_\varepsilon(0)} f(x-y)\, dS(y).$$ Changing variables $x-y=z$

and recalling that $\mathrm{vol}(\partial B_\varepsilon(0)) = \mathrm{vol}(\partial B_\varepsilon(x)) = n\alpha(n)\varepsilon^{n-1}$ we obtain

$$-\frac{1}{\mathrm{vol}(\partial B_\varepsilon(x))}\int_{\partial B_\varepsilon(x)} f(z)\, dS(z) \longrightarrow -f(x) \quad \text{as } \varepsilon\to 0$$ by fact 3.

Putting all together we have

$$\Delta u(x) = \lim_{\varepsilon\to 0}\left[\int_{B_\varepsilon(0)} \Gamma(y)\Delta_x f(x-y)\, dy + \int_{\mathbb{R}^n\setminus B_\varepsilon(0)} \Gamma(y)\Delta_x f(x-y)\, dy\right] = -f(x),$$

finishing the proof. $\boxed{\diagdown}$