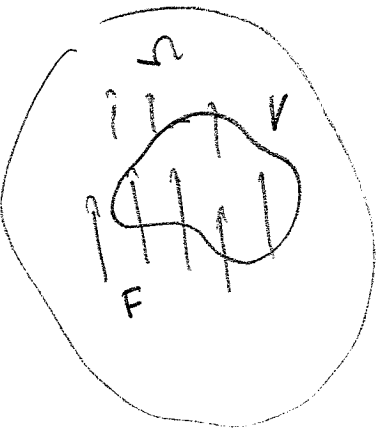


## Laplace's equation and Poisson's equation

We will investigate Laplace's equation:  $\Delta u = 0$  and Poisson's equation  $-\Delta u = f$ , both in a domain  $\Omega \subseteq \mathbb{R}^n$ . In both cases the unknown is  $u = u(x_1, \dots, x_n)$ . The negative sign on  $-\Delta u$  is purely conventional, and solving  $-\Delta u = f$  is of course equivalent to solving  $\Delta u = \tilde{f}$ ,  $\tilde{f} = -f$ .

Laplace's equation appears in many applications. For example, suppose we have the density of some quantity  $u$  (e.g. a chemical concentration) in equilibrium in  $\Omega$ . If  $V$  is any subregion of  $\Omega$ , the net flux of  $u$  through  $\partial V$  must be zero. Denoting the flux by  $F = (F_1, F_2, \dots, F_n)$  we have

$$\int_{\partial V} F \cdot \nu \, ds = 0.$$
 Typically, the flux is proportional to the gradient of  $u$  but points in the opposite direction (the flow is from regions of higher to lower concentration):  $F = -k \nabla u$ ,



Using the divergence theorem:

$$0 = \int_{\partial V} \mathbf{E} \cdot \mathbf{v} \, dS = \int_V \operatorname{div} \mathbf{E} \, dx = \int_V -k \operatorname{div} \nabla u \, dx, \quad \text{but}$$

$$\operatorname{div} \nabla u = \operatorname{div} \left( \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right) = \frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_n^2} = \Delta u, \quad \text{thus,}$$

$$\int_V \Delta u \, dx = 0. \quad \text{Since this must hold for any region } V, \text{ we find that:}$$

$$\Delta u = 0 \quad \text{in } \mathbb{R}^n.$$

Solutions to Laplace's equation can also be thought as stationary solutions to the heat or wave equations.

Poisson's equation also appears in many applications. For instance, from Maxwell's equations  $\operatorname{curl} \mathbf{E} = 0$  and  $\operatorname{div} \mathbf{E} = \rho$ ,  $\rho = \text{charge density}$ , we have (from  $\operatorname{curl} \mathbf{E} = 0$ ) that  $\mathbf{E} = -\nabla \phi$ , where  $\phi$  is known as the electric potential. Thus  $\operatorname{div} \mathbf{E} = -\operatorname{div} \nabla \phi = \rho$  i.e.,

$$-\Delta \phi = \rho$$

One important property of Laplace's equation is that it is invariant under rotations, meaning the following. A  $n \times n$  matrix  $M$  is called an orthogonal matrix if  $MM^T = I$ , where  $M^T$  is the transpose of  $M$  and  $I$  is the identity matrix (thus, orthogonal matrices are always invertible and  $M^{-1} = M^T$ ). If  $x \in \mathbb{R}^n$ , we think of  $Mx$  as the vector obtained from  $x$  by rotating it about some axis (which depends on  $M$ ). For instance, a rotation of  $\theta$  about the  $z$ -axis in  $\mathbb{R}^3$  is

given by

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Suppose that  $u$  solves  $\Delta u = 0$ . Define a new function  $\tilde{u}$  by  $\tilde{u}(x) = u(Mx)$ . Then  $\Delta \tilde{u} = 0$ .

Since rotations do not destroy the property of being a solution to Laplace's equation, we attempt to solve it by seeking rotationally symmetric solutions.

Take  $\Omega = \mathbb{R}^n$ , set  $u(x) = \sigma(r)$  where  $r = |x| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$ .

We will try to select  $\sigma$  so that it solves Laplace's equation. Note that

$$\frac{\partial r}{\partial x_i} = \frac{1}{2} \frac{2x_i}{\sqrt{x_1^2 + \dots + x_n^2}} = \frac{x_i}{r} \quad \text{provided that } r \neq 0.$$

$$\text{Then } \frac{\partial u}{\partial x_i} = \frac{\partial \sigma}{\partial r} \frac{\partial r}{\partial x_i} = \sigma' \frac{x_i}{r} \quad (\sigma' = \frac{\partial \sigma}{\partial r})$$

$$\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial}{\partial x_i} \left( \sigma' \frac{x_i}{r} \right) = \frac{\partial \sigma'}{\partial x_i} \frac{x_i}{r} + \sigma' \frac{\partial}{\partial x_i} \left( \frac{x_i}{r} \right) = \frac{\partial \sigma'}{\partial r} \frac{\partial r}{\partial x_i} \frac{x_i}{r} + \sigma' \frac{r \frac{\partial x_i}{\partial x_i} - x_i \frac{\partial r}{\partial x_i}}{r^2}$$

$$= \sigma'' \frac{x_i^2}{r^2} + \frac{\sigma'}{r^2} \left( r - \frac{x_i^2}{r} \right) = \sigma'' \frac{x_i^2}{r^2} + \sigma' \left( \frac{1}{r} - \frac{x_i^2}{r^3} \right). \quad \text{Therefore:}$$

$$\begin{aligned} \Delta u &= \frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_n^2} = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = \sum_{i=1}^n \left( \sigma'' \frac{x_i^2}{r^2} + \sigma' \left( \frac{1}{r} - \frac{x_i^2}{r^3} \right) \right) \\ &= \frac{\sigma''}{r^2} \sum_{i=1}^n x_i^2 + \sum_{i=1}^n \frac{\sigma'}{r} - \frac{\sigma'}{r^3} \sum_{i=1}^n x_i^2 = \frac{\sigma''}{r^2} r^2 + n \frac{\sigma'}{r} - \frac{\sigma'}{r^3} r^2 = \sigma'' + \frac{n-1}{r} \sigma' \end{aligned}$$

ie, 
$$\Delta u = \sigma'' + \frac{n-1}{r} \sigma'$$

Since we want  $\Delta u = 0$ , we need to solve the following ODE:

$$\sigma'' + \frac{n-1}{r} \sigma' = 0. \quad \text{If } \sigma' \neq 0, \text{ this can be written as } \frac{\sigma''}{\sigma'} = \frac{1-n}{r}.$$

But  $\frac{\sigma''}{\sigma'} = (\ln|\sigma'|)'$ , so  $(\ln|\sigma'|)' = \frac{1-n}{r}$ . Integrating both sides

gives  $\ln|\sigma'| = \int \frac{1-n}{r} dr = (1-n) \ln r + A = A + \ln r^{1-n}$ , so  $|\sigma'| = e^A e^{\ln r^{1-n}}$

We can remove the absolute value and absorb the  $\pm$  sign in the constant  $e^A$ , which we relabel  $a$ , so  $\sigma' = \frac{a}{r^{n-1}}$ . We integrate again to find  $\sigma$ :

$$\sigma(r) = \begin{cases} b \ln r + c & \text{if } n=2 \\ \frac{b}{r^{n-2}} + c & \text{if } n \geq 3 \end{cases} \quad \text{where } b \text{ and } c \text{ are constants.}$$

Note that  $u(x) = \sigma(r)$  is not quite a solution to Laplace's equation in  $\mathbb{R}^n$ , since it is not defined at  $x=0$  ( $r=0$ ). However, we will see how it can be used to construct actual solutions.

We will make a specific choice for the constants  $b$  and  $c$  which will be convenient in the ensuing calculations.

Def. The function  $\Gamma: \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$  given by

$$\Gamma(x) = \begin{cases} -\frac{1}{2\pi} \ln |x|, & n=2 \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}}, & n \geq 3 \end{cases}$$

is called the Fundamental Solution to Laplace's equation

Here,  $\alpha(n)$  is the volume of the ball of radius one and center zero in  $\mathbb{R}^n$  (so  $\alpha(3) = \frac{4}{3}\pi$ ).

We are going to need the quantities  $\partial_i \Gamma$  and  $\partial_i \partial_j \Gamma$ . Let's compute, for  $n \geq 3$ :

$$\begin{aligned} \partial_i \Gamma(x) &= \frac{1}{n(n-2)\alpha(n)} \partial_i |x|^{2-n} = \frac{2-n}{n(n-2)\alpha(n)} |x|^{1-n} \underbrace{\partial_i |x|}_{= \partial_i r = \frac{x_i}{r} = \frac{x_i}{|x|}} \\ &= -\frac{1}{n\alpha(n)} \frac{x_i}{|x|^n} \end{aligned}$$

$$\partial_j \partial_i \Gamma(x) = -\frac{1}{n \alpha(n)} \partial_j \frac{x_i}{|x|^n} = -\frac{1}{n \alpha(n)} \frac{|x|^n \partial_j x_i - x_i \partial_j |x|^n}{|x|^{2n}}. \quad \text{We have that}$$

$$\partial_j x_i = \begin{cases} 0, & j \neq i \\ 1, & j = i \end{cases}, \quad \text{thus we introduce the symbol } \delta_{ij} \text{ defined by } \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

so that  $\partial_j x_i = \delta_{ij}$ .  $\partial_j |x|^n = n |x|^{n-1} \partial_j |x| = n |x|^{n-1} \frac{x_j}{|x|} = n |x|^{n-2} x_j$ . Hence:

$$\begin{aligned} \partial_j \partial_i \Gamma(x) &= -\frac{1}{n \alpha(n)} \frac{1}{|x|^{2n}} \left( |x|^n \delta_{ij} - n |x|^{n-2} x_i x_j \right) \\ &= -\frac{1}{n \alpha(n)} \left( |x|^2 \delta_{ij} - n x_i x_j \right) \frac{1}{|x|^{n+2}} \end{aligned}$$

Summary up

$$\partial_i \Gamma(x) = -\frac{1}{n \alpha(n)} \frac{x_i}{|x|^n} \quad \text{and} \quad \partial_i \partial_j \Gamma(x) = -\frac{1}{n \alpha(n)} \left( |x|^2 \delta_{ij} - n x_i x_j \right) \frac{1}{|x|^{n+2}}$$

We are going to also need estimates for  $\partial_i \Gamma$  and  $\partial_i \partial_j \Gamma$ . For this,

observe that 
$$\left| \frac{x_i}{|x|^n} \right| = \frac{|x_i|}{|x|^n} = \frac{\sqrt{x_i^2}}{|x|^n} \leq \frac{\sqrt{x_1^2 + \dots + x_n^2}}{|x|^n} = \frac{1}{|x|^{n-1}} \quad \text{and}$$

$$\left| \frac{x_i x_j}{|x|^{n+2}} \right| = \frac{|x_i| |x_j|}{|x|^{n+2}} = \frac{\sqrt{x_i^2} \sqrt{x_j^2}}{|x|^{n+2}} \leq \frac{\sqrt{x_1^2 + \dots + x_n^2} \sqrt{x_1^2 + \dots + x_n^2}}{|x|^{n+2}} = \frac{1}{|x|^n}$$

Thus

$$|\partial_i \Gamma(x)| \leq \frac{1}{n \alpha(n)} \frac{1}{|x|^{n-1}}, \quad \text{and}$$

$$\begin{aligned} |\partial_i \partial_j \Gamma(x)| &= \frac{1}{n \alpha(n)} \left| \frac{|x|^2}{|x|^{n+2}} \delta_{ij} - n \frac{x_i x_j}{|x|^{n+2}} \right| \leq \frac{1}{n \alpha(n)} \left( \frac{\delta_{ij}}{|x|^n} + n \frac{|x_i x_j|}{|x|^{n+2}} \right) \\ &\leq \frac{1}{n \alpha(n)} \left( \frac{1}{|x|^n} + \frac{n}{|x|^n} \right) = \frac{n+1}{n \alpha(n)} \frac{1}{|x|^n} \end{aligned}$$

Summing up:

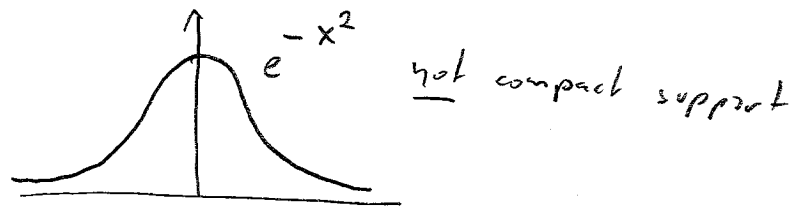
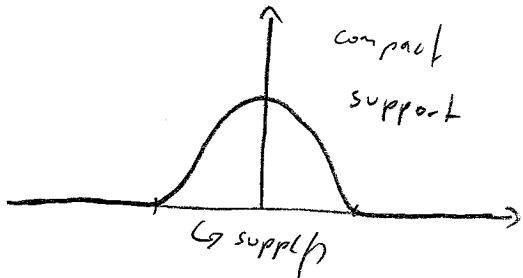
$$|\partial_i \Gamma(x)| \leq \frac{1}{n \alpha(n)} \frac{1}{|x|^{n-1}}, \quad |\partial_i \partial_j \Gamma(x)| \leq \frac{n+1}{n \alpha(n)} \frac{1}{|x|^n}$$

Analogous estimates hold in  $\mathbb{R}^2$ .



Def. Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  and set  $A = \{x \in \mathbb{R}^n \mid f(x) \neq 0\}$ . We define the support of  $f$ , denoted  $\text{supp}(f)$ , as the set  $\text{supp}(f) = \bar{A}$ . If  $\bar{A} \subseteq B_R(0)$  (= ball of radius  $R$  and center zero) for some  $R$ , we say that  $f$  has compact support. The space of  $C^k$  functions with compact support is denoted by  $C_c^k(\mathbb{R}^n)$ .

Remark. If  $f \in C_c^k(\mathbb{R}^n)$ , then  $f(x) = 0$  for all  $x$  such that  $|x|$  is sufficiently large.



Theorem (solution to Poisson's equation) Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be a  $C^2$  function with compact support. Set

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y) f(y) \quad (\text{i.e., } u = \Gamma * f) \quad \text{Then}$$

(i)  $u$  is well-defined

(ii)  $u \in C^2(\mathbb{R}^n)$

(iii)  $-\Delta u = f$ .