HOMEWORK 9 SOLUTIONS - HEAT EQUATION PROJECT

MATH 3120

In this assignment, you will guided to construct solutions to the initial-value problem for the heat equation in $\mathbb{R}^n$:

$$u_t - \Delta u = 0 \text{ in } (0,\infty) \times \mathbb{R}^n. \tag{1}$$

Although this assignment has the complexity of a small class project, in that it is longer than a typical HW and stands as a self-contained topic, it will be graded as a regular homework.

Unless stated otherwise, the notation below is as in class.

Question 1. Look for a solution to (1) in the form

$$u(t,x) = t^{-\alpha} v(t^{-\beta} x), \tag{2}$$

where α and β will be chosen and v will be determined. More precisely, proceed as follows:

(a) Show that plugging (2) into (1) produces

$$\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y \cdot \nabla v(y) + t^{-(\alpha+2\beta)} \Delta v(y) = 0, \tag{3}$$

where $y := t^{-\beta} x$.

(b) Set $\beta = \frac{1}{2}$ in (3) to obtain

$$\Delta v(y) + \frac{1}{2} y \cdot \nabla v(y) + \alpha v(y) = 0. \tag{4}$$

(c) Assume that v is radially symmetric, i.e.,

$$v(y) = w(r), \tag{5}$$

where w is to be determined. Show that in this case (4) becomes

$$w'' + \frac{n-1}{r} w' + \frac{1}{2} r w' + \alpha w = 0. \tag{6}$$

(d) Set $\alpha = \frac{n}{2}$ in (6) to find

$$(r^{n-1} w')' + \frac{1}{2}(r^n w)' = 0. \tag{7}$$

(e) From (7), conclude that

$$r^{n-1} w' + \frac{1}{2} r^n w = A, \tag{8}$$

where A is a constant.

(f) Set $A = 0$ in (8) and conclude that

$$w(r) = B e^{-\frac{1}{4} r^2}, \tag{9}$$

where B is a constant.

(g) Combine (2), (5), (9), and take into account the choices of α and β, to conclude that

$$u(t,x) = \frac{B}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}},\ t > 0, \tag{10}$$

is a solution to (1).

Solution 1. These are simply a sequence of straightforward calculations.

The previous question motivates the following definition. The function

$$\Gamma(t,x) := \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, & t > 0, x \in \mathbb{R}^n, \\ 0, & t < 0, x \in \mathbb{R}^n, \end{cases}$$

is called the *fundamental solution of the heat equation.* Note that for $t > 0$, $\Gamma(t,x)$ is simply (10) with a specific choice of the constant B. This choice of B is to guarantee Γ to integrate to 1 (see the next question). In particular, $\Gamma(t,x)$ is a solution of (1).

Question 2. Use the fact that

$$\int_{\mathbb{R}^n} e^{-|x|^2}\, dx = \pi^{\frac{n}{2}} \tag{11}$$

to show that for each $t > 0$

$$\int_{\mathbb{R}^n} \Gamma(t,x)\, dx = 1.$$

(You do *not* have to show (11).)

Solution 2. Set $z = x/\sqrt{4t}$ and change variables to find

$$\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}}\, dx = \int_{\mathbb{R}^n} e^{-|z|^2} (\sqrt{4t})^n\, dz = \pi^{\frac{n}{2}} (4t)^{\frac{n}{2}}.$$

We now consider the initial-value problem for the heat equation:

$$u_t - \Delta u = 0, \qquad \text{in } (0,\infty) \times \mathbb{R}^n, \tag{12a}$$

$$u(0,x) = g(x),\ \ x \in \mathbb{R}^n. \tag{12b}$$

Define

$$u(t,x) := \int_{\mathbb{R}^n} \Gamma(t, x-y) g(y)\, dy,\ t > 0, x \in \mathbb{R}^n. \tag{13}$$

For the next questions, in (12), assume that $g \in C^0(\mathbb{R}^n)$ and that there exists a constant $C > 0$ such that $|g(x)| \le C$ for all $x \in \mathbb{R}^n$.

Question 3. Show that (13) is well-defined.

Solution 3. We have

$$|u(t,x)| \le \frac{C}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}\, dy.$$

Making the change of variables $z = (y-x)/\sqrt{4t}$ we find

$$\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}\, dy = (4t)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-|z|^2}\, dz < \infty.$$

Question 4. Show that $u \in C^\infty((0,\infty)\times\mathbb{R}^n)$, where u is defined by (13).
Hint: Use the following fact, that you do *not* need to prove. Let α be a multiindex and $t > 0$. If

$$\int_{\mathbb{R}^n} D_x^\alpha \Gamma(t, x-y) g(y)\, dy$$

is well-defined, then

$$D^\alpha u(t,x) = \int_{\mathbb{R}^n} D_x^\alpha \Gamma(t, x-y) g(y)\, dy,$$

where we write D_x^α on the RHS to emphasize that the differentiation is with respect to the x variable.

Solution 4. Let $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$ be an arbitrary multiindex. Then

$$D_x^\alpha \Gamma(t, x-y) = \frac{p(t,x,y)}{t^M} e^{-\frac{|x-y|^2}{4t}}, \tag{14}$$

where M is a non-negative constant and p is a polynomial on its arguments (If (14) is not clear, take a few derivatives of $\Gamma(t, x-y)$ and see the pattern that emerges.) Then, using the assumption on g,

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} D_x^\alpha \Gamma(t,x-y) g(y)\, dy \right| &\le C \int_{\mathbb{R}^n} |D_x^\alpha \Gamma(t,x-y)|\, dy \\
&\le C \int_{\mathbb{R}^n} \frac{|p(t,x,y)|}{t^M} e^{-\frac{|x-y|^2}{4t}}\, dy \\
&= \int_{\mathbb{R}^n} \frac{|q(t,x,z)|}{t^N} e^{-|z|^2}\, dz,
\end{aligned}$$

where in the last step we changed variables $z = (y-x)/\sqrt{4t}$, N is a non-negative constant, and q is polynomial on its arguments. We claim that there exists a constant $C > 0$, possibly depending on t, such that

$$\frac{|q(t,x,z)|}{t^N} e^{-|z|^2} \le C e^{-\frac{1}{2}|z|^2}. \tag{15}$$

For, (15) is equivalent to

$$\frac{|q(t,x,z)|}{t^N} e^{-\frac{1}{2}|z|^2} \le C. \tag{16}$$

For each fixed x and $t > 0$, the function $\frac{|q(t,x,z)|}{t^N} e^{-\frac{1}{2}|z|^2}$ is a continuous function of z, and because the exponential decays faster than any polynomial, we conclude that $\frac{|q(t,x,z)|}{t^N} e^{-\frac{1}{2}|z|^2}$ is bounded in $\mathbb{R}^n$ as a function of z for each fixed x and $t > 0$, which is (16). Since the integral of $e^{-\frac{1}{2}|z|^2}$ is finite, we have shown the result in view of the hint and the fact that α, x, and $t > 0$ are arbitrary.

Question 5. Show that u given by (13) is a solution to the initial-value problem (12).
Hint: Use the following fact, that you do *not* need to prove. For each $x_0 \in \mathbb{R}^n$,

$$\lim_{(t,x)\to(0,x_0)} u(t,x) = g(x_0).$$

Solution 5. By construction,

$$\partial_t \Gamma(t, x-y) - \Delta_x \Gamma(t, x-y) = 0$$

for $t > 0$. Thus, differentiating under the integral sign (which we can in view of the discussion of the previous problem) and using the hint, we immediately obtain the result.

Question 6. In (12), assume further that g has compact support and that $g \geq 0$. Show that for any $t > 0$ and any $x \in \mathbb{R}^n$, $u(t, x) \neq 0$. Explain why this can be interpreted as saying that, for the heat equation, information propagates at infinite speed. Contrast it with the finite speed of propagation for the wave equation.

Solution 6. From

$$u(t, x) = \int_{\mathbb{R}^n} \Gamma(t, x-y) g(y)\, dy,\ t > 0, x \in \mathbb{R}^n.$$

and the assumptions, we conclude that for any $t > 0$ and any $x \in \mathbb{R}^n$, we have $u(t, x) > 0$. Thus, even though u starts off compactly supported, it immediately (i.e., for any arbitrarily small $t > 0$) becomes positive at any x. Clearly, this can only happen if "information propagates at infinity speed."