

## HOMWORK 8 SOLUTIONS

MATH 3120

Unless stated otherwise, the notation below is as in class.

**Question 1.** Use Duhamel's principle to show that a solution to the inhomogeneous wave equation in  $1d$  with zero data and source term  $f$  is given by

$$u(t, x) = \frac{1}{2} \int_0^t \int_{x-s}^{x+s} f(t-s, y) dy ds. \quad (1)$$

To do so, first use D'Alembert's formula to conclude that

$$u_s(t, x) = \frac{1}{2} \int_{x-t+s}^{x+t-s} f(s, y) dy.$$

Use the definition of  $u$  in terms of  $u_s$  and change variables to conclude (1).

**Solution 1.** Using D'Alembert's formula, we find

$$u_s(t, x) = \frac{1}{2} \int_{x-t+s}^{x+t-s} f(s, y) dy,$$

where we used the fact that D'Alembert's formula was derived for data at  $t = 0$ ; for data at  $t = s$  we have to replace  $t$  by  $t - s$  in the limits of integration. Thus

$$u(t, x) = \frac{1}{2} \int_0^t \int_{x-t+s}^{x+t-s} f(s, y) dy ds = \frac{1}{2} \int_0^t \int_{x-z}^{x+z} f(t-z, y) dy dz,$$

where we made the change  $s = t - z$ .

**Question 2.** Use Duhamel's principle to show that a solution to the inhomogeneous wave equation in  $3d$  with zero data and source term  $f$  is given by

$$u(t, x) = \frac{1}{4\pi} \int_{B_t(x)} \frac{f(t - |y - x|, y)}{|y - x|} dy. \quad (2)$$

(The integrand in (2) is known as the retarded potential.) To do so, first use Kirchhoff's formula for solutions in  $n = 3$  to conclude that

$$u_s(t, x) = \frac{t-s}{\text{vol}(\partial B_{t-s}(x))} \int_{\partial B_{t-s}(x)} f(s, y) dS(y).$$

Use the definition of  $u$  in terms of  $u_s$  and change variables to conclude (2).

**Solution 2.** Kirchhoff's formula gives

$$u_s(t, x) = \frac{1}{\text{vol}(\partial B_{t-s}(x))} \int_{\partial B_{t-s}(x)} (t-s) f(s, y) dS(y).$$

Thus

$$\begin{aligned}
 u(t, x) &= \int_0^t \frac{t-s}{\text{vol}(\partial B_{t-s}(x))} \int_{\partial B_{t-s}(x)} f(s, y) dS(y) ds \\
 &= \frac{1}{4\pi} \int_0^t \int_{\partial B_{t-s}(x)} \frac{f(s, y)}{t-s} dS(y) ds \\
 &= \frac{1}{4\pi} \int_0^t \int_{\partial B_r(x)} \frac{f(t-r, y)}{r} dS(y) dr \\
 &= \frac{1}{4\pi} \int_{B_t(x)} \frac{f(t-|y-x|, y)}{|y-x|} dy,
 \end{aligned}$$

where we made the change of variables  $r = t - s$  and then wrote  $r = |y - x|$ .

**Question 3.** Show that there exists a constant  $C > 0$  such that for any solution  $u$  to the 3d wave equation it holds that

$$|u(t, x)| \leq \frac{C}{t} \int_{\mathbb{R}^3} (|D^2 u_0(y)| + |Du_0(y)| + |u_0(y)| + |Du_1(y)| + |u_1(y)|) dy$$

for  $t \geq 1$ .

*Hint:* Use Kirchoff's formula, note that for any function  $f$  we have

$$f(y) = f(y) \frac{y-x}{t} \cdot \frac{y-x}{t}$$

on  $\partial B_t(x)$ , and use one of Green's identities.

**Solution 3.** We have

$$u(t, x) = \frac{1}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x)} (u_0(y) + tu_1(y) + \nabla u_0(y) \cdot (y-x)) dS(y).$$

The unit outer normal to  $\partial B_t(x)$  is  $\nu = (y-x)/t$ , so that  $\nu \cdot \nu = \frac{y-x}{t} \cdot \frac{y-x}{t} = 1$ . Therefore, using this and Green's identities,

$$\begin{aligned}
 \frac{1}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x)} u_0(y) dS(y) &= \frac{1}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x)} u_0(y) \nu \cdot \frac{y-x}{t} dS(y) \\
 &= \frac{1}{\text{vol}(\partial B_t(x))} \int_{B_t(x)} \text{div}_y \left( u_0(y) \frac{y-x}{t} \right) dy \\
 &= \frac{1}{\text{vol}(\partial B_t(x))} \int_{B_t(x)} \left( \nabla u_0(y) \cdot \frac{y-x}{t} + u_0(y) \frac{3}{t} \right) dy,
 \end{aligned}$$

so that

$$\begin{aligned}
 \left| \frac{1}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x)} u_0(y) dS(y) \right| &\leq \frac{C}{t^2} \int_{B_t(x)} (|\nabla u_0(y)| + |u_0(y)|) dy \\
 &\leq \frac{C}{t^2} \int_{\mathbb{R}^3} (|\nabla u_0(y)| + |u_0(y)|) dy.
 \end{aligned}$$

A similar inequality holds for the  $u_1$  integral (with an extra factor of  $t$ ), and for  $\nabla u_0$ :

$$\begin{aligned} \frac{1}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x)} \nabla u_0(y) \cdot (y-x) dS(y) &= \frac{t}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x)} \nabla u_0(y) \cdot \nu dS(y) \\ &= \frac{1}{4\pi t} \int_{B_t(x)} \Delta u_0(y) dy, \end{aligned}$$

so that

$$\left| \frac{1}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x)} \nabla u_0(y) \cdot (y-x) dS(y) \right| \leq \frac{C}{t} \int_{\mathbb{R}^3} |D^2 u_0(y)| dy.$$

Combining the foregoing produces the result.