

HOMEWORK 7 SOLUTIONS

MATH 3120

Unless stated otherwise, the notation below is as in class.

Question 1. Prove the differentiation of moving regions formula stated in class:

$$\frac{d}{d\tau} \int_{\Omega(\tau)} f \, dx = \int_{\Omega(\tau)} \partial_\tau f \, dx + \int_{\partial\Omega(\tau)} f v \cdot \nu \, dS. \quad (1)$$

(See the class notes for the notation and precise assumptions.) For simplicity, prove (1) in the following particular case. Assume that $n = 3$ and that the domains $\Omega(\tau)$ are given by a one-parameter family of one-to-one and onto maps $\varphi = \varphi(\tau, x) : \Omega \rightarrow \Omega(\tau) = \varphi(\tau, \Omega)$, where $\Omega := \Omega(0)$ and $\varphi(0, \cdot) = \text{id}_\Omega$, where id_Ω is the identity map on Ω , i.e., $\text{id}_\Omega(x) = x$, $x \in \Omega$.

(a) For each fixed τ , consider the change of variables $x = \varphi(\tau, y)$, so that

$$\int_{\Omega(\tau)} f(\tau, x) \, dx = \int_{\Omega} f(\tau, \varphi(\tau, y)) J(\tau, y) \, dy, \quad (2)$$

where $J(\tau, y)$ is the Jacobian of the transformation $x = \varphi(\tau, y)$ for fixed τ .

(b) Show that there exists a one parameter family of vector fields $u(\tau, \cdot)$ such that

$$\partial_\tau \varphi(\tau, x) = u(\tau, \varphi(\tau, x)).$$

(c) Explain why $u = v$ on $\partial\Omega(\tau)$.

(d) Show that

$$\partial_\tau J(\tau, x) = (\text{div } u)(\tau, \varphi(\tau, x)) J(\tau, x).$$

(e) Use (2) and the above to compute $\frac{d}{d\tau} \int_{\Omega(\tau)} f$, and do an integration by parts to obtain the result.

Solution 1. (a) This is simply the change of variables formula from calculus.

(b) For each fixed x , the map $\tau \mapsto \varphi(\tau, x)$ is a curve in \mathbb{R}^3 . $\partial_\tau \varphi(\tau, x)$ is, therefore, the tangent vector to this curve at $\varphi(\tau, x)$ at time τ . The collection of all such tangent vectors, as τ and x vary, forms the vector field u .

(c) The map φ sends $\partial\Omega$ onto $\partial\Omega(\tau)$ for each τ . Since $\partial_\tau \varphi(\tau, x)$ is the velocity at time τ of the particle that started at $x \in \Omega$ at time zero, $u(\tau, \varphi(\tau, x))$ is the velocity of $\partial\Omega(\tau)$ at the point $\varphi(\tau, x) \in \partial\Omega(\tau)$.

(d) According to the notation of part (a), we set

$$\varphi_j^i = \frac{\partial}{\partial y^j} \varphi^i, \quad \partial_j u^i = \frac{\partial}{\partial x^j} u^i,$$

where we considered $\varphi = (\varphi^1, \varphi^2, \varphi^3)$. In particular, note that when we write $\varphi_j^i = \partial_j \varphi^i$ the derivative is always with respect to $y \in \Omega$, whereas when we write $\partial_j u^i$ the derivative is always with respect to $x \in \Omega(\tau)$.

Recall the following formula for the determinant of a $n \times n$ matrix a with entries $a_j^i = a_{\text{column}}^{\text{row}}$:

$$\det(a) = \frac{1}{n!} \epsilon_{i_1 \dots i_n} \epsilon^{j_1 \dots j_n} a_{j_1}^{i_1} \dots a_{j_n}^{i_n}.$$

In our case, this gives

$$J(\tau, y) = \frac{1}{3!} \epsilon_{i_1 i_2 i_3} \epsilon^{j_1 j_2 j_3} \varphi_{j_1}^{i_1} \varphi_{j_2}^{i_2} \varphi_{j_3}^{i_3}.$$

Recall that the definition of J involves an absolute value, which we can omit here since $J > 0$ because $J(0, \cdot) > 0$. Compute

$$\begin{aligned} \partial_\tau \varphi_j^i &= \partial_j \partial_\tau \varphi^i \\ &= \frac{\partial}{\partial y^j} u^i \\ &= \partial_\ell u^i \varphi_j^\ell, \end{aligned}$$

where in the second equality we used (b) and in the third one the chain rule. Therefore

$$\partial_\tau J(\tau, y) = \frac{1}{3!} \epsilon_{i_1 i_2 i_3} \epsilon^{j_1 j_2 j_3} (\partial_\ell u^{i_1} \varphi_{j_1}^\ell \varphi_{j_2}^{i_2} \varphi_{j_3}^{i_3} + \varphi_{j_1}^{i_1} \partial_\ell u^{i_2} \varphi_{j_2}^\ell \varphi_{j_3}^{i_3} + \varphi_{j_1}^{i_1} \varphi_{j_2}^{i_2} \partial_\ell u^{i_3} \varphi_{j_3}^\ell). \quad (3)$$

Because $\epsilon_{i_1 i_2 i_3}$ is non-zero only for $i_1 i_2 i_3$ all different from each other, for each triple $i_1 i_2 i_3$, the term $\epsilon_{i_1 i_2 i_3} \partial_\ell u^{i_1} \varphi_{j_1}^\ell \varphi_{j_2}^{i_2} \varphi_{j_3}^{i_3}$ is non-zero only when $\ell = i_1$. Similarly for the second and third terms on the RHS of (3), and we obtain

$$\partial_\tau J(\tau, y) = \frac{1}{3!} \sum_{\substack{i_1, i_2, i_3=1 \\ j_1, j_2, j_3=1}}^3 \epsilon_{i_1 i_2 i_3} \epsilon^{j_1 j_2 j_3} (\partial_{i_1} u^{i_1} + \partial_{i_2} u^{i_2} + \partial_{i_3} u^{i_3}) \varphi_{j_1}^{i_1} \varphi_{j_2}^{i_2} \varphi_{j_3}^{i_3}.$$

Because the summand is non-zero only if $i_1 i_2 i_3$ are all different from each other, the term in parenthesis is always equal to $\partial_1 u^1 + \partial_2 u^2 + \partial_3 u^3 = \operatorname{div} u$, which gives the result.

(e) We have

$$\begin{aligned}
\frac{d}{d\tau} \int_{\Omega(\tau)} f \, dx &= \partial_\tau \int_{\Omega} f(\tau, \varphi(\tau, y)) J(\tau, y) \, dy \\
&= \int_{\Omega} (\partial_\tau f(\tau, \varphi(\tau, y)) J(\tau, y) + \nabla f(\tau, \varphi(\tau, y)) \cdot \partial_\tau \varphi(\tau, y) J(\tau, y) + f(\tau, \varphi(\tau, y)) \partial_\tau J(\tau, y)) \, dy \\
&= \int_{\Omega} (\partial_\tau f(\tau, \varphi(\tau, y)) J(\tau, y) + \nabla f(\tau, \varphi(\tau, y)) \cdot u(\tau, \varphi(\tau, y)) J(\tau, y) \\
&\quad + f(\tau, \varphi(\tau, y)) (\operatorname{div} u)(\tau, \varphi(\tau, y)) J(\tau, y)) \, dy \\
&= \int_{\Omega(\tau)} (\partial_\tau f(\tau, x) + \nabla f(\tau, x) \cdot u(\tau, x) + f(\tau, x) (\operatorname{div} u)(\tau, x)) \, dx \\
&= \int_{\Omega(\tau)} (\partial_\tau f(\tau, x) - f(\tau, x) (\operatorname{div} u)(\tau, x) + f(\tau, x) (\operatorname{div} u)(\tau, x)) \, dx \\
&\quad + \int_{\partial\Omega(\tau)} f(\tau, x) u(\tau, x) \cdot \nu(\tau, x) \, dS(x) \\
&= \int_{\Omega(\tau)} \partial_\tau f(\tau, x) \, dx + \int_{\partial\Omega(\tau)} f(\tau, x) v(\tau, x) \cdot \nu(\tau, x) \, dS(x).
\end{aligned}$$

Above, we the steps are as follows: in the second line we used the product rule and the chain rule; in the third line we used (b) and (d); on the fourth line, we changed variables back to x ; on the fifth line we integrated ∇f by parts (equivalently, used one of the Green identities); on the last line, we used (c).

Question 2. Let u be a solution to the Cauchy problem for the wave equation in \mathbb{R}^n . Assume that u_0 and u_1 have their supports in the ball $B_R(0)$ for some $R > 0$. Show that $u = 0$ in the exterior of the region

$$I := \{(t, x) \in (0, \infty) \times \mathbb{R}^n \mid x \in B_{R+t}(0)\}.$$

I is called a domain of influence for that data on $B_R(0)$ (compare with the 1d case).

Solution 2. Let $(t, x) \notin I$. Then $K_{t,x}^- \cap I = \emptyset$, and the result follows from the finite-propagation speed for the wave equation.

Question 3. Let u be a solution to the Cauchy problem for the wave equation and assume that u_0 and u_1 have compact support.

(a) Show that the energy

$$E(t) := \frac{1}{2} \int_{\mathbb{R}^n} [(\partial_t u)^2 + |\nabla u|^2] \, dx$$

is well-defined.

(b) Show that

$$E(t) = E(0),$$

i.e., the energy is conserved.

Solution 3. (a) By question 2, the solution u has compact support for each fixed t .

(b) For each t_0 and $\varepsilon > 0$, there exists, by (a), a $R_* > 0$ such that $u(t, x) = 0$ for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ and $|x| \geq R_*$. We now follow the proof of the finite-propagation speed property for the wave equation (see the class notes) using the ball B_{R_*} , and observe the following. In that proof, we did an integration by parts, and controlled the boundary term using the Cauchy-Schwarz inequality. Here, this boundary term vanishes identically by the foregoing. We obtain therefore a sequence of equalities (rather than inequalities as in the proof done in class), which then gives the result.

Question 4. Let u be a solution to the Cauchy problem for the wave equation in \mathbb{R}^3 with compactly supported data (i.e., u_0 and u_1 have compact support).

(a) Show that there exists a constant $C > 0$, depending on u_0 and u_1 , such that

$$|u(t, x)| \leq \frac{C}{t}, \quad (4)$$

for $t \geq 1$ and $x \in \mathbb{R}^3$. Thus, for each fixed x , u approaches zero as $t \rightarrow \infty$, i.e., solutions decay in time.

Hint: Use the formula for solutions in $n = 3$. Since the data has compact support, it vanishes outside $B_R(0)$ for some $R > 0$. This implies an estimate for the area of the largest region within $B_t(x)$ where the data is non-trivial.

(b) Is the estimate (4) sharp? (I.e., can it be improved to show that solutions decay faster in time than $\frac{1}{t}$?)

(c) Do we still get decay if the data does not have compact support?

Solution 4. (a) The solution is given by

$$u(t, x) = \frac{1}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x)} (u_0(y) + tu_1(y) + \nabla u_0(y) \cdot (y - x)) dS(y).$$

Since the data is compactly supported, there exists a $R > 0$ such that $u_0(x) = 0$ and $u_1(x) = 0$ for $|x| \geq R$, so that

$$u(t, x) = \frac{1}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_R(0)} (u_0(y) + tu_1(y) + \nabla u_0(y) \cdot (y - x)) dS(y).$$

Because the data is compactly supported, we have $|u_0|, |u_1|, |\nabla u_0| \leq C$ for some $C > 0$, so that

$$\begin{aligned} |u(t, x)| &\leq \frac{C}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_R(0)} (1 + t + |y - x|) dS \\ &= \frac{C}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_R(0)} \left(1 + t + \frac{t|y - x|}{t}\right) dS \\ &\leq \frac{C}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_R(0)} (1 + t + t) dS \\ &\leq \frac{C(1 + t)}{t^2} \int_{\partial B_t(x) \cap B_R(0)} dS, \end{aligned}$$

where we used that $|y - x|/t = 1$ since $y \in B_t(x)$ and that $\text{vol}(\partial B_t(x)) = 4\pi t^2$. Because $\partial B_t(x) \cap B_R(0)$ has area at most $4\pi R^2$, we have the result.

(b) Yes, it cannot be improved for arbitrary solutions of the wave equation. To see this, take $u_0 = 0$ and u_1 to be a non-negative compactly supported function that is equal to 1 on $B_1(0)$. Then

$$\begin{aligned} u(t, x) &= \frac{1}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x)} t u_1(y) dS(y) \\ &= \frac{t}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_1(0)} u_1(y) dS(y) + \frac{t}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x) \setminus (B_t(x) \cap B_1(0))} u_1(y) dS(y). \end{aligned}$$

Note that the second term on the RHS is always non-negative, thus

$$u(t, x) \geq \frac{t}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_1(0)} u_1(y) dS(y) = \frac{t}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_1(0)} dS.$$

For any x on the boundary of the lightcone, i.e., $|x| = t$, and such that $|x| \geq 1$, we have that the area of $\partial B_t(x) \cap B_1(0)$ is $\geq C > 0$, so that $u(t, x) \geq C/t$.

(c) Not necessarily, e.g., take $u_0 = 0$ and $u_1 = 1$, then $u(t, x) = t$ is the solution.