HOMEWORK 6 SOLUTIONS

MATH 3120

Unless stated otherwise, the notation below is as in class.

Question 1. Prove the following fact that we used in the construction of solutions to Poisson's equation: let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous, then

$$\lim_{r\to 0^+} \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} f\, dS = f(x).$$

Hint: Consider the difference $f(x) - \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} f\, dS$ and use $\frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} dS = 1$.

Solution 1. We have to prove that given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < r < \delta$ then

$$\left| \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} f\, dS - f(x) \right| < \varepsilon.$$

Write

$$\begin{aligned}\frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} f(y)\, dS(y) - f(x) &= \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} f(y)\, dS - \frac{f(x)}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} dS(y) \\ &= \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} (f(y) - f(x))\, dS(y).\end{aligned}$$

Thus

$$\left| \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} f(y)\, dS(y) - f(x) \right| \leq \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} |f(y) - f(x)|\, dS(y).$$

Fix $\varepsilon > 0$. Since f is continuous, there exists a $\delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. If $r < \delta$, then $|x - y| < \delta$ for all $y \in \partial B_r(x)$, thus

$$\left| \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} f(y)\, dS(y) - f(x) \right| < \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} \varepsilon\, dS = \varepsilon.$$

Question 2. In class, we constructed solutions to Poisson's equation in $\mathbb{R}^n$ for $n \geq 3$. Carry out the construction in the case $n = 2$. You do *not* have to do all the steps. Rather, follow what was done in class and point out what changes in $n = 2$. This boils down to slightly modifying some of the estimates for the fundamental solution.

Solution 2. We use the following estimates in the $n = 2$ case:

$$\int_{B_\varepsilon(0)} |\Gamma(y)|\, dy \leq C\varepsilon^2 |\ln \varepsilon| \to 0 \text{ as } \varepsilon \to 0^+,$$

and

$$\int_{\partial B_\varepsilon(0)} |\Gamma(y)|\, dS(y) \leq C\varepsilon |\ln \varepsilon| \to 0 \text{ as } \varepsilon \to 0^+,$$

and the rest of the proof is essentially the same.

Question 3. Let u be a non-trivial harmonic function in $\mathbb{R}^n$. Can u have compact support? *Hint:* mean value theorem.

Solution 3. No. Let u be harmonic and with compact support and fix an arbitrary $x \in \mathbb{R}^n$. By the compact support property, there exists a $r > 0$ such that $u(y) = 0$ for all $y \in \partial B_r(x)$. By the mean value formula

$$u(x) = \frac{1}{\mathrm{vol}(\partial B_r(x))} \int_{\partial B_r(x)} u(y)\, dS(y) = 0,$$

so that $u = 0$ since x is arbitrary.

Question 4. Prove the converse of the mean value theorem. I.e., let $u \in C^2(\Omega)$ be such that

$$u(x) = \frac{1}{\mathrm{vol}(\partial B_r(x))} \int_{\partial B_r(x)} u\, dS$$

for each $B_r(x) \subset \Omega$. Show that $\Delta u = 0$ in Ω.
Hint: Assume that $\Delta u(x) \neq 0$ for some $x \in \Omega$. Use the functions $f(r)$, $f'(r)$ used in the proof of the mean value to derive a contradiction.

Solution 4. If u is not harmonic, there exists a $x \in \Omega$ such that $\Delta u(x) \neq 0$. By assumption, the function

$$f(r) = \frac{1}{\mathrm{vol}(\partial B_r(x))} \int_{\partial B_r(x)} u\, dS$$

is constant equal to $u(x)$ on the interval $(0, R)$, where $R > 0$ is a fixed number such that $B_R(x) \subset \Omega$. Thus $f'(r) = 0$ for all $r \in (0, R)$. On the other hand, by continuity, Δu has a definite sign, say positive, on a ball $B_{r_0}(x)$ for some $r_0 > 0$, which without loss of generality we can take such that $r_0 < R$. Arguing as in the proof of the mean value theorem, we find

$$f'(r_0) = \frac{1}{n\omega_n r_0^{n-1}} \int_{B_{r_0}(x)} \Delta u(y)\, dy > 0,$$

contradicting $f'(r_0) = 0$.

Question 5. Let Γ be the fundamental solution to the Laplacian in $\mathbb{R}^n$. Compute

$$\partial_i |x|,$$

and use this to show that there exists a constant $C > 0$ such that

$$|\partial_i \Gamma| \leq \frac{C}{|x|^{n-1}},\ |\partial^2_{ij}\Gamma| \leq \frac{C}{|x|^n},\ i,j = 1,\dots,n,\ x \neq 0.$$

Solution 5. We have

$$\begin{aligned}\partial_i |x| &= \partial_i \sqrt{(x^1)^2 + \cdots + (x^n)^2} \\ &= \frac{x^i}{\sqrt{(x^1)^2 + \cdots + (x^n)^2}} \\ &= \frac{x^i}{|x|}.\end{aligned}$$

Thus,

$$\begin{aligned}
\partial_i \frac{1}{|x|^{n-2}} &= (2-n)\frac{1}{|x|^{n-1}}\partial_i |x| \\
&= (2-n)\frac{1}{|x|^{n-1}}\frac{x^i}{|x|} \\
&= (2-n)\frac{x^i}{|x|^n},
\end{aligned}$$

$$\begin{aligned}
\partial_j \partial_i \frac{1}{|x|^{n-2}} &= (2-n)\partial_j \frac{x^i}{|x|^n} \\
&= (2-n)\frac{\delta^i_j}{|x|^n} + (n-2)n\frac{x^i x^j}{|x|^{n+2}}.
\end{aligned}$$

The result for $n \geq 3$ follows by observing that

$$\left|\frac{x^i}{|x|}\right| \leq 1.$$

The calculations for $n = 2$ are similar.