HOMEWORK 5 SOLUTIONS

MATH 3120

Unless stated otherwise, the notation below is as in class.

Question 1. Consider the following initial-value problem for the wave equation in one dimension:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \text{ in } (-\infty, \infty) \times (0, \infty), \\ u(x,0) &= f(x), \\ u_t(x,0) &= g(x), \end{aligned} \tag{1}$$

(a) Solve (1) when $f(x) = x^2$ and $g(x) = 0$.

(b) Assume now that $c = 1$ and

$$f(x) = \begin{cases} 1, & -2 \le x \le 0 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$g(x) = \begin{cases} -1, & -1 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Draw a diagram in the (x,t)-plane indicating the different regions where the solution is influenced by the initial condition f and g and the regions where the solution is identically zero. (You do **not** have to find u.)

Solution 1. (a) By D'Alembert's formula:

$$u(x,t) = \frac{(x+ct)^2 + (x-ct)^2}{2}.$$

(b) The regions are summarized in the Figure 1.

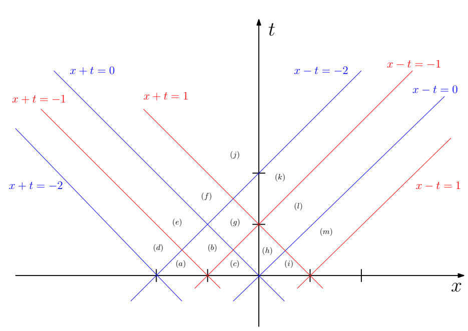

FIGURE 1. Problem 1b.

Question 2. Write each PDE below in the form $F(x, u, Du, \dots, D^m u) = 0$, i.e., identify the function F. State if the PDE is homogeneous or non-homogeneous, linear or non-linear.

(a) $u_{tt} - u_{xx} = f$.

(b) $u_y + u u_x = 0$.

(c) $a^{ijk} \partial^3_{ijk} v + v = 0$,

where i, j, k range from 1 to 3.

(d) $u_{xx} + x^2 y^2 u_{yy} = (x + y)^2$.

(e) $u_{xy} + \cos(u) = \sin(xy)$.

Solution 2. In order to find F, it is useful to identify whether the PDE is linear, homogeneous, the unknown function, etc.

(a) Unknown: u. Independent variables: x, t. Order: second. We have

$$F(p_1, \dots, p_9) = p_9 - p_6 - f(p_1, p_2).$$

The equation is linear and non-homogeneous.

(b) Unknown: u. Independent variables: x, y. Order: first. We have

$$F(p_1, \dots, p_5) = p_5 + p_3 p_4.$$

The equation is non-linear (because of the term $u u_x$) and homogeneous.

(c) It is instructive to consider a slightly more general case, with i, j, k ranging from 1 to n. Unknown: v. Independent variables: $x^1, \dots, x^n$. Order: third. We have

$$F(x_1, \dots, x_n, p, p_1, \dots, p_n, p_{11}, \dots, p_{nn}, \dots, p_{111}, \dots, p_{nnn}) = a^{ijk} p_{ijk} + p.$$

The equation is linear and homogeneous.

(d) Unknown: u. Independent variables: x, y. Order: second. We have

$$F(p_1, \dots, p_9) = p_6 + p_1^2 p_2^2 p_9 - (p_1 + p_2)^2.$$

The equation is linear and non-homogeneous.

(e) Unknown: u. Independent variables: x, y. Order: second. We have

$$F(p_1, \dots, p_9) = p_7 + \cos p_3 - \sin(p_1 p_2).$$

The equation is non-linear (because of $\cos u$) and non-homogeneous.

Question 3. Consider a PDE $F(x, u, Du, \dots, D^m u) = 0$. Prove that the PDE is linear (as defined in class in terms of linearity of F with respect to some of its entries) if and only if it can be written as

$$\sum_{|\alpha| \leq k} a_\alpha D^\alpha u = f,$$

as stated in class.

Solution 3. Denote by F_H the homogeneous part of F.

Suppose the PDE is linear. Thus,

$$F_H(x, u, Du, \ldots, D^m u) = \sum_{k=0}^{m} F_k(x, D^k u), \tag{2}$$

where each F_k is a sum of linear functions on derivatives of u of order k, i.e.,

$$F_k(x, D^k u) = \sum_{\ell=1}^{n^k} F_{k\ell}(x, u^{(\ell)}), \tag{3}$$

where each $u^{(\ell)}$ represents one of the n^k possible derivatives of u of order k. Let u and v be two functions for which $F(x, u, Du, \ldots, D^m u)$ and $F(x, v, Dv, \ldots, D^m v)$ are well-defined, but are otherwise arbitrary, and let a and b be two arbitrary constants. Then

$$F_k(x, aD^k u + bD^k v) = a \sum_{\ell=1}^{n^k} F_{k\ell}(x, u^{(\ell)}) + b \sum_{\ell=1}^{n^k} F_{k\ell}(x, v^{(\ell)})$$

by the linearity of $F_{k\ell}$. Hence

$$F_H(x, au + bv, aDu + bv, \ldots, aD^m u + bD^m v) = aF_H(x, u, Du, \ldots, D^m u) + bF_H(x, v, Dv, \ldots, D^m v).$$

Writing for simplicity $Pu = F_H(x, u, Du, \ldots, D^m u)$, we conclude

$$P(au + bv) = aF_H(x, u, Du, \ldots, D^m u) + bF_H(x, v, Dv, \ldots, D^m v) = aPu + bPv,$$

as desired.

Reciprocally, suppose that P is a linear operator. Then it can be written on the form

$$\begin{aligned} Pu = {} & a^{i_1 i_2 \cdots i_m} \partial^m_{i_1 i_2 \cdots i_m} u + a^{i_1 i_2 \cdots i_{m-1}} \partial^{m-1}_{i_1 i_2 \cdots i_{m_1}} u \\ & + a^{i_1 i_2 \cdots i_{m-2}} \partial^{m-1}_{i_1 i_2 \cdots i_{m_2}} u + \cdots + a^{i_1 i_2} \partial^2_{i_1 i_2} u + a^i \partial_i u + au. \end{aligned}$$

This implies that F_H has the decomposition (2) with each F_k satisfying (3).