# HOMEWORK 4 SOLUTIONS

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

**Question 1.** The goal of this problem is to prove the following theorem stated in class: Let $g, h \in C^2([0,L])$ satisfy $g(0) = g(L) = 0 = h(0) = h(L)$ and $g''(0) = g''(L) = 0 = h''(0) = h''(L)$. Then, the formal solution

$$u(t,x) = \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi ct}{L} + b_n \sin \frac{n\pi ct}{L} \right) \sin \frac{n\pi x}{L},$$

where $a_n$ and $b_n$ are given by

$$a_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx,$$
$$b_n = \frac{2}{n\pi c} \int_0^L h(x) \sin \frac{n\pi x}{L}\, dx,$$

is a $C^2$ solution of the initial-boundary value problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, && \text{in } (0,\infty) \times (0,L), \\ u(t,0) = u(t,L) &= 0, && t \geq 0, \\ u(0,x) &= g(x), && 0 \leq x \leq L, \\ \partial_t u(0,x) &= h(x), && 0 \leq x \leq L, \end{aligned}$$

To prove the theorem, proceed as follows.

(a) Show that $g$ and $h$ can be extended to $2L$-periodic $C^2$ odd functions on $\mathbb{R}$. Call these extensions $\tilde{g}$ and $\tilde{h}$.

(b) Use D'Alembert's formula to solve the initial value problem for the wave equation on $\mathbb{R}$ with data $\tilde{g}$ and $\tilde{h}$. (In class we derived D'Alembert's formula with $c = 1$; here you need the formula for a general $c$.)

(c) Consider the Fourier series for $\tilde{g}$ and $\tilde{h}$. Plug these into D'Alembert's formula and using trigonometric identities arrive at the expression given by the formal solution for $x \in [0, L]$. Observe that the boundary conditions are satisfied.

(d) In all the above, make sure that you have the correct assumptions to guarantee the convergence of the Fourier series you employ and whatever other theorem you may need to invoke.

**Solution 1.** Set $\tilde{g}(x) = g(x)$ if $0 \leq x \leq L$, $\tilde{g}(x) = -g(-x)$ if $-L \leq x < 0$, and extend it periodically with period $2L$. Construct $\tilde{h}$ similarly. Then, because of $g(0) = g(L) = 0 = h(0) = h(L)$ and $g''(0) = g''(L) = 0 = h''(0) = h''(L)$, $\tilde{g}, \tilde{h} \in C^2(\mathbb{R})$.

The solution to the Cauchy problem for the wave equation with data $(\tilde{g}, \tilde{h})$ is

$$\tilde{u}(t,x) = \frac{\tilde{g}(x+ct) + \tilde{g}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{h}(y)\,dy.$$

Since $\tilde{g}$ is $C^2$, its Fourier series converges to $\tilde{g}$ pointwise. Thus,

$$\tilde{g}(y) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi y}{L} + b_n \sin\frac{n\pi y}{L}\right),$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} \tilde{g}(x)\cos\frac{n\pi x}{L}\,dx = 0,$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} \tilde{g}(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} g(x)\sin\frac{n\pi x}{L}\,dx,$$

since $\tilde{g}$ is odd and $\tilde{g}(x) = g(x)$ for $x \in [0, L]$. Then

$$\begin{aligned}
\frac{\tilde{g}(x+ct) + \tilde{g}(x-ct)}{2} &= \frac{1}{2}\sum_{n=1}^{\infty}\left(b_n \sin\frac{n\pi(x+ct)}{L} + b_n \sin\frac{n\pi(x-ct)}{L}\right) \\
&= \frac{1}{2}\sum_{n=1}^{\infty} b_n \Big( \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} + \sin\frac{n\pi ct}{L}\cos\frac{n\pi x}{L} \\
&\quad + \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} - \sin\frac{n\pi ct}{L}\cos\frac{n\pi x}{L}\Big) \\
&= \sum_{n=1}^{\infty} b_n \cos\frac{n\pi ct}{L}\sin\frac{n\pi x}{L}.
\end{aligned}$$

Similarly,

$$\tilde{h}(y) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi y}{L} + B_n \sin\frac{n\pi y}{L}\right),$$

where

$$A_n = \frac{1}{L}\int_{-L}^{L} \tilde{h}(x)\cos\frac{n\pi x}{L}\,dx = 0,$$

and

$$B_n = \frac{1}{L}\int_{-L}^{L} \tilde{h}(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} h(x)\sin\frac{n\pi x}{L}\,dx,$$

since $\tilde{h}$ is odd and $\tilde{h}(x) = h(x)$ for $x \in [0, L]$. Then

$$\begin{aligned}
\frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{h}(y)\,dy &= \frac{1}{2c}\int_{x-ct}^{x+ct}\sum_{n=1}^{\infty} B_n \sin\frac{n\pi y}{L}\,dy \\
&= \frac{1}{2c}\sum_{n=1}^{\infty} B_n \int_{x-ct}^{x+ct}\sin\frac{n\pi y}{L}\,dy
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2c}\sum_{n=1}^{\infty}\frac{B_n L}{n\pi}\left(\cos\frac{n\pi(x+ct)}{L} - \cos\frac{n\pi(x-ct)}{L}\right) \\
&= -\frac{1}{2c}\sum_{n=1}^{\infty}\frac{B_n L}{n\pi}\Big(\cos\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} - \sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L} \\
&\quad - \cos\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} - \sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}\Big) \\
&= \frac{1}{2}\sum_{n=1}^{\infty}\frac{B_n L}{nc\pi}\sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L},
\end{aligned}
$$

where we could integrate term-by-term because $\tilde{h}$ is $C^2$. Plugging these expressions back into D'Alembert's formula we obtain the result.

**Question 2.** In class we saw that if $u_0 \in C^2(\mathbb{R})$ and $u_1 \in C^1(\mathbb{R})$, then the Cauchy problem for the $1d$ wave equation with data $(u_0, u_1)$ admits a unique $C^2$ solution. What can you say if $u_0 \in C^k(\mathbb{R})$ and $u_0 \in C^{k-1}(\mathbb{R})$, $k > 2$?

**Solution 2.** According to D'Alembert's formula, there will be a unique $C^k$ solution.

**Question 3.** In class we solved the $1d$ wave equation for $t \geq 0$. Making a change of variables $t \mapsto -t$, show that we can also solve the wave equation for negative times. Conclude that D'Alembert's formula is valid for $-\infty < t < \infty$.

**Solution 3.** This follows from showing that the wave equation is invariant under $t \mapsto -t$.