HOMEWORK 3 SOLUTIONS

MATH 3120

The notation and terminology below is the same used in class.

Question 1. Find the Fourier series of the given functions:

(a) $f(x) = x$, $-1 \leq x \leq 1$.

(b) $f(x) = \sin(5x)$, $-\pi \leq x \leq \pi$.

Solution 1. (a) We have $L = 1$. Compute

$$a_n = \int_{-1}^{1} x\cos(n\pi x)\,dx = 0,$$
$$b_n = \int_{-1}^{1} x\sin(n\pi x)\,dx = \frac{2(-1)^{n+1}}{n\pi}.$$

Therefore

$$F.S.\{f\}(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(n\pi x).$$

(a) We have $L = \pi$. Compute

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(5x)\cos(\frac{n\pi x}{\pi})\,dx = 0,$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(5x)\sin(\frac{n\pi x}{\pi})\,dx = \begin{cases} 0, & n \neq 5, \\ 1, & n = 5. \end{cases}$$

Therefore

$$F.S.\{f\}(x) = \sin(5x) = f(x).$$

Try to understand the meaning of this result in linear algebra terms.

Question 2. Consider the function

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Show that $f \in C^0(\mathbb{R})$, that f is differentiable, but $f \notin C^1(\mathbb{R})$.

Solution 2. f is C^∞ for any $x \neq 0$. Using the squeeze theorem, we verify that f is continuous at zero. To find f' at zero, we compute

$$f'(0) = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h} = \lim_{h\to 0}\frac{h^2\sin\frac{1}{h}}{h} = \lim_{h\to 0} h\sin\frac{1}{h} = 0.$$

For $x \neq 0$,

$$f'(x) = (x^2 \sin \frac{1}{x})' = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

Since $\lim_{x \to 0} f'(x)$ does not exist, we have that $\lim x \to 0 f'(x) \neq f'(0)$ and f' is not continuous.

Question 3.

(a) Prove that $C^k(I)$ is a vector space.

(b) Prove that the derivative $\frac{d}{dx}$ is a linear map between $C^k(I)$ and $C^{k-1}(I)$. What happens in the case $k = \infty$?

Solution 3. (a) Linear combinations of functions that are k-times continuously differentiable are again k-times continuously differentiable; $0 \in C^k(I)$ (where 0 means the identically zero function).

(b) The derivative is linear and if f is k-times continuously differentiable then f' is $(k-1)$-times continuously differentiable. If $k = \infty$, then $f' \in C^\infty$ for any $f \in C^\infty$, thus in this case $\frac{d}{dx}$ is a linear map form C^∞ to itself.