HOMEWORK 2 SOLUTIONS

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

Question 1. Show that in spherical coordinates the Laplacian reads

$$\Delta = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta_{S^2},$$

where

$$\Delta_{S^2} := \partial_\phi^2 + \frac{\cos\phi}{\sin\phi}\partial_\phi + \frac{1}{\sin^2\phi}\partial_\theta^2$$

is the Laplacian on the unit sphere, $r \in [0, \infty)$, $\phi \in [0, \pi]$, and $\theta \in [0, 2\pi)$.

Solution 1. Recall that the relation between Cartesian coordinates (x, y, z) and spherical coordinates (r, ϕ, θ) is

$$\begin{aligned} x &= r\sin\phi\cos\theta, \\ y &= r\sin\phi\sin\theta, \\ z &= r\cos\phi, \end{aligned}$$

and

$$r^2 = x^2 + y^2 + z^2, \tag{1}$$

$$\sin\phi = \frac{\sqrt{x^2+y^2}}{r}, \tag{2}$$

$$\tan\theta = \frac{y}{x}. \tag{3}$$

From (1) we obtain

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r} = \sin\phi\cos\theta.$$

From (2) we obtain

$$\frac{\partial\phi}{\partial x} = \frac{x\cos^2\phi}{z^2\tan\phi} = \frac{\cos\phi\cos\theta}{r}.$$

From (3) we obtain

$$\frac{\partial\theta}{\partial x} = -\frac{\sin\theta}{r\sin\phi}.$$

Similarly, we compute the derivatives with respect to y and z, finding

$$\begin{aligned} \frac{\partial r}{\partial y} &= \sin\phi\sin\theta, \\ \frac{\partial\phi}{\partial y} &= \frac{\cos\phi\sin\theta}{r}, \end{aligned}$$

$$\begin{aligned}
\frac{\partial\theta}{\partial y} &= \frac{\cos\theta}{r\sin\phi},\\
\frac{\partial r}{\partial z} &= \cos\phi,\\
\frac{\partial\phi}{\partial z} &= -\frac{\sin\phi}{r},\\
\frac{\partial\theta}{\partial z} &= 0.
\end{aligned}$$

Therefore, using the chain rule and the above,

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}\\
&= \sin\phi\cos\theta\frac{\partial}{\partial r} + \frac{\cos\phi\cos\theta}{r}\frac{\partial}{\partial\theta} - \frac{\sin\theta}{r\sin\phi}\frac{\partial}{\partial\theta}.
\end{aligned}$$

Next, we compute,

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \frac{\partial^2}{\partial x^2} = \sin\phi\cos\theta\Big[&\sin\phi\cos\theta\frac{\partial^2}{\partial r^2} + \frac{\cos\phi\cos\theta}{r}\frac{\partial^2}{\partial r\partial\phi}\\
&- \frac{\cos\phi\cos\theta}{r^2}\frac{\partial}{\partial\phi} - \frac{\sin\theta}{r\sin\phi}\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin\theta}{r^2\sin\phi}\frac{\partial}{\partial\theta}\Big]\\
&+ \frac{\cos\phi\cos\theta}{r}\Big[\sin\phi\cos\theta\frac{\partial^2}{\partial r\partial\phi} + \cos\phi\cos\theta\frac{\partial}{\partial r}\\
&\frac{\cos\phi\cos\theta}{r}\frac{\partial^2}{\partial\phi^2} - \frac{\sin\phi\cos\theta}{r}\frac{\partial}{\partial\phi} - \frac{\sin\theta}{r\sin\phi}\frac{\partial^2}{\partial\theta\partial\phi}\\
&\frac{\cos\phi\sin\theta}{r\sin^2\phi}\frac{\partial}{\partial\theta}\Big] - \frac{\sin\theta}{r\sin\phi}\Big[\sin\phi\cos\theta\frac{\partial^2}{\partial r\partial\theta}\\
&- \sin\phi\sin\theta\frac{\partial}{\partial r} + \frac{\cos\phi\cos\theta}{r}\frac{\partial^2}{\partial\theta\partial\phi} - \frac{\cos\phi\sin\theta}{r}\frac{\partial}{\partial\phi}\\
&- \frac{\sin\theta}{r\sin\phi}\frac{\partial^2}{\partial\theta^2} - \frac{\cos\theta}{r\sin\phi}\frac{\partial}{\partial\theta}\Big].
\end{aligned}$$

Similarly we compute $\frac{\partial^2}{\partial y^2}$ and $\frac{\partial^2}{\partial z^2}$. We then plug these expressions into Δ and observe that there are several cancellations. This then gives the desired result.

In the next two questions, you will be asked to perform some of the computations skipped in class in our derivation of solutions to the Schödinger equation for an electrostatic potential.

Question 2. Consider the PDE

$$\frac{d}{dx}\left((1-x^2)\frac{d\Phi}{dx}\right) + \left(\lambda - \frac{m^2}{1-x^2}\right)\Phi = 0. \tag{4}$$

Show that a solution to (4) is given by

$$\Phi(x) = (1-x^2)^{\frac{|m|}{2}}\frac{d^{|m|}P(x)}{dx^{|m|}}, \tag{5}$$

where P solves

$$(1-x^2)\frac{d^2P}{dx^2}-2x\frac{dP}{dx}+\lambda P=0. \tag{6}$$

Hint: Differentiate (6) a few times and observe the resulting pattern, or use induction, to conclude that if P solves (6) then

$$(1-x^2)\frac{d^{|m|+2}P}{dx^{|m|+2}}-2(|m|+1)x\frac{d^{|m|+1}P}{dx^{|m|+1}}+(\lambda-|m|(|m|+1))\frac{d^{|m|}P}{dx^{|m|}}=0. \tag{7}$$

Next, let $\widetilde{\Phi}$ be defined by

$$\Phi(x)=(1-x^2)^{\frac{|m|}{2}}\widetilde{\Phi}(x), \tag{8}$$

where Φ is a solution to (4). Plugging (8) into (4), conclude that $\widetilde{\Phi}$ satisfies

$$(1-x^2)\frac{d^2\widetilde{\Phi}}{dx^2}-2x\left(|m|+1\right)\frac{d\widetilde{\Phi}}{dx}+\left(\lambda-|m|(|m|+1)\right)\widetilde{\Phi}=0. \tag{9}$$

Compare (10) with (7) to obtain the result.

Solution 2. Equation (7) can be easily derived upon differentiation of (6).

Plugging (8) into (4) we find

$$\begin{aligned}
\frac{d}{dx}\left((1-x^2)\frac{d}{dx}\left((1-x^2)^{\frac{|m|}{2}}\widetilde{\Phi}\right)\right) &= \frac{d}{dx}\left(\frac{|m|}{2}(-2x)(1-x^2)^{\frac{|m|}{2}}\widetilde{\Phi}+(1-x^2)^{\frac{|m|}{2}+1}\frac{d\widetilde{\Phi}}{dx}\right)\\
&=(1-x^2)^{\frac{|m|}{2}+1}\frac{d^2\widetilde{\Phi}}{dx^2}+(1-x^2)^{\frac{|m|}{2}}\frac{d\Phi}{dx}\left(\left(\frac{|m|}{2}+1\right)(-2x)+\frac{|m|}{2}(-2x)\right)\\
&+\frac{|m|}{2}\left((-2x)\frac{|m|}{2}(1-x^2)^{\frac{|m|}{2}-1}(-2x)-2(1-x^2)^{\frac{|m|}{2}}\right)\widetilde{\Phi}\\
&=(1-x^2)^{\frac{|m|}{2}+1}\frac{d^2\widetilde{\Phi}}{dx^2}-2x(1-x^2)^{\frac{|m|}{2}}\left(|m|+1\right)\frac{d\widetilde{\Phi}}{dx}+\frac{|m|}{2}(1-x^2)^{\frac{|m|}{2}}\left(\frac{2|m|x^2}{1-x^2}-2\right)\widetilde{\Phi}\\
&=(1-x^2)^{\frac{|m|}{2}}\left((1-x^2)\frac{d^2\widetilde{\Phi}}{dx^2}-2x\left(|m|+1\right)\frac{d\widetilde{\Phi}}{dx}+|m|\left(\frac{|m|x^2}{1-x^2}-1\right)\widetilde{\Phi}\right).
\end{aligned}$$

By (4), this has to equal

$$-\left(\lambda-\frac{m^2}{1-x^2}\right)\Phi=\left(\lambda-\frac{m^2}{1-x^2}\right)(1-x^2)^{\frac{|m|}{2}}\widetilde{\Phi}(x),$$

what gives, after canceling $(1-x^2)^{\frac{|m|}{2}}$,

$$(1-x^2)\frac{d^2\widetilde{\Phi}}{dx^2}-2x\left(|m|+1\right)\frac{d\widetilde{\Phi}}{dx}+\lambda\widetilde{\Phi}+\left(\frac{|m|^2x^2}{1-x^2}-|m|-\frac{|m|^2}{1-x^2}\right)\widetilde{\Phi}=0.$$

But

$$\frac{|m|^2x^2}{1-x^2}-|m|-\frac{|m|^2}{1-x^2}=\frac{|m|(|m|+1)x^2-|m|(|m|+1)}{1-x^2}=-|m|(|m|+1),$$

and therefore

$$(1-x^2)\frac{d^2\widetilde{\Phi}}{dx^2}-2x\left(|m|+1\right)\frac{d\widetilde{\Phi}}{dx}+\left(\lambda-|m|(|m|+1)\right)\widetilde{\Phi}=0. \tag{10}$$

Comparing (10) with (7), we see that if P solves (6), then (8) solves (4), as claimed.

Question 3. Consider the PDE

$$\frac{1}{\varrho^2}\frac{d}{d\varrho}\left(\varrho^2\frac{dR}{d\varrho}\right)+\left(-\frac{1}{4}-\frac{\ell(\ell+1)}{\varrho^2}+\frac{\gamma}{\varrho}\right)R=0. \tag{11}$$

Show that (11) does not admit a non-trivial solution of the form

$$R(\varrho)=\sum_{k=0}^{\infty}a_k\varrho^k. \tag{12}$$

Hint: Plug (12) into (11) to derive

$$\begin{aligned}&-\ell(\ell+1)a_0\varrho^{-2}+((2-\ell(\ell+1))a_1+\gamma a_0)\,\varrho^{-1}\\&+\sum_{k=0}^{\infty}\left(((k+3)(k+2)-\ell(\ell+1))\,a_{k+2}+\gamma a_{k+1}-\frac{1}{4}a_k\right)\varrho^k=0.\end{aligned} \tag{13}$$

From (13), conclude that $a_k=0$ for all k.

Solution 3. After deriving (13), we immediately see that if $\ell\neq 0$ (recall that $\ell=0,1,2,\dots$), then all coefficients must vanish. If $\ell=0$, we obtain a recurrence relation that implies that the series does not converge for all ϱ unless all coefficients are zero.

Question 4. Consider the following initial-boundary value problem for the wave equation in one dimension:

$$\begin{aligned}u_{tt}-c^2u_{xx}&=0, && \text{in } (0,\infty)\times(0,L),\\ u(t,0)=u(t,L)&=0, && t\geq 0,\\ u(0,x)&=g(x), && 0\leq x\leq L,\\ \partial_t u(0,x)&=h(x), && 0\leq x\leq L,\end{aligned}$$

where g and h are given and satisfy the compatibility conditions $g(0)=g(L)=0=h(0)=h(L)$. Use separation of variables to show that a formal solution to this problem is given by

$$u(t,x)=\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi ct}{L}+b_n\sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L},$$

where a_n and b_n are arbitrary coefficients.

Solution 4. Assuming that $u(t,x)=T(t)X(x)$ and plugging into the wave equation gives

$$\frac{T''}{c^2T}=\frac{X''}{X}.$$

Since the LHS depends only on t and the RHS only on x, this implies that $\frac{T''}{c^2T}=\lambda=\frac{X''}{X}$, where λ is a constant. The boundary conditions imply that $X(0)=X(L)=0$.

The solution to $X''-\lambda X=0$ depends on the sign of λ. For $\lambda>0$ or $\lambda=0$, the solution is a combination of exponential functions or a polynomial, respectively. In both cases the boundary conditions cannot be satisfied unless X is identically zero. For $\lambda<0$, we find that $X(x)=c_1\cos(\sqrt{-\lambda}x)+c_2\sin(\sqrt{-\lambda}x)$, where c_1 and c_2 are constants. $X(0)=0$ implies $c_1=0$. $X(L)=0$ implies that $c_2=0$ or $\sin(\sqrt{-\lambda}L)=0$; the former option would give an identically zero solution, thus we must have $\sqrt{-\lambda}L=n\pi$, i.e., $\lambda=-\frac{n^2\pi^2}{L^2}$, $n=1,2,3,\dots$.

Using this into $T'' - c^2\lambda T = 0$ and solving for T gives the result. (As seen in class, the constants a_n and b_n are arbitrary as solution to the equation with boundary conditions, but they are determined by the initial condition as Fourier coefficients.)

Question 5. Consider the following initial-boundary value problem for the heat equation in one dimension:

$$\begin{aligned} u_t - u_{xx} &= 0, && \text{in } (0,\infty) \times (0, L), \\ u(t,0) = u(t,L) &= 0, && t \geq 0, \\ u(0,x) &= g(x), && 0 \leq x \leq L, \end{aligned}$$

where g is given and satisfies the compatibility conditions $g(0) = g(L) = 0$. Use separation of variables to show that a formal solution to this problem is given by

$$u(t,x) = \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2}{L^2}t} \sin \frac{n\pi x}{L},$$

where the a_n's are arbitrary coefficients. What happens when $t \to \infty$? Interpret your answer physically.

Solution 5. The derivation of the formal solution is similar to the previous problem. Assuming convergence of the series and that we can exchange the infinite sum and the limit $t \to \infty$, we obtain that u converges to zero as t goes to infinite. This problem models the temperature in the region $[0, L]$ with the extremities kept at a fixed temperature equal to zero, thus we expect that the temperature in the interior will eventually reach zero as well.