HOMEWORK 12 SOLUTIONS

MATH 3120

Question 1. Consider the Cauchy problem for Burgers' equation with data given by

$$h(x) = \begin{cases} 1, & x \leq 0, \\ 1-x, & 0 < x < 1, \\ 0, & x \geq 1. \end{cases}$$

(a) Show that the solution is given by

$$u(t,x) = \begin{cases} 1, & x \leq t, t < 1, \\ \frac{1-x}{1-t}, & t < x < 1, t < 1, \\ 0, & x \geq 1, t < 1. \end{cases}$$

(b) The denominator of $\frac{1-x}{1-t}$ approaches zero when $t \to 1^-$. Does that mean that $|u(t,x)| \to \infty$ as $t \to 1^-$? Does this not contradict your result from question 4 in HW 11? What exactly is becoming singular when the characteristics intersect at $(1,1)$?
(c) Let $0 < \beta < 1$ and define, for $t \geq 1$

$$\tilde{u}(t,x) = \begin{cases} 1, & x < \beta t + 1 - \beta, \\ 0 & x > \beta t + 1 - \beta. \end{cases}$$

Show that v given by

$$v(t,x) = \begin{cases} u(t,x), & 0 \leq t < 1, \\ \tilde{u}(t,x), & t \geq 1, \end{cases}$$

is a weak solution if and only if $\beta = 1/2$. (This was essentially done in class. Here, you have to work out the calculations in more detail, including the case when Ω might intersect the region $\{t \leq 1\}$.)

Solution 1. (a) By a direct calculation, we see that u is, for $t < 1$, a classical solution except along the lines $x = t$ and $x = 1$. Let φ be a test function with support within Ω', with $\Omega' \cap \{t \geq 1\} = \varnothing$, and set $\Omega := \Omega' \cap \{t \geq 0\}$. Put $\Omega_1 := \Omega \cap \{x < t\}$, $\Omega_2 := \Omega \cap \{t < x < 1\}$, and $\Omega_3 := \Omega \cap \{x > 1\}$. Then, using our assumptions and integration by parts, we find

$$\int_0^\infty \int_{-\infty}^\infty (\partial_t \varphi u + \partial_x \varphi F(u))\, dxdt + \int_{-\infty}^\infty \varphi(0,x) u(0,x)\, dx = \int_\Omega (\partial_t \varphi u + \partial_x \varphi F(u))\, dxdt$$
$$+ \int_{\Omega\cap\{t=0\}} \varphi(0,x)u(0,x)\, dx$$
$$= \int_{\Omega_1} (\partial_t \varphi u + \partial_x \varphi F(u))\, dxdt + \int_{\Omega_2} (\partial_t \varphi u + \partial_x \varphi F(u))\, dxdt + \int_{\Omega_3} (\partial_t \varphi u + \partial_x \varphi F(u))\, dxdt$$
$$+ \int_{\Omega_1\cap\{t=0\}} \varphi(0,x)u(0,x)\, dx + \int_{\Omega_2\cap\{t=0\}} \varphi(0,x)u(0,x)\, dx + \int_{\Omega_3\cap\{t=0\}} \varphi(0,x)u(0,x)\, dx$$
$$= \int_{\{x=t\}\cap\overline{\Omega}_1} \varphi(u\nu_1^t + F(u)\nu_1^x)\, ds + \int_{\{x=t\}\cap\overline{\Omega}_2} \varphi(u\nu_2^t + F(u)\nu_2^x)\, ds$$
$$+ \int_{\{x=1\}\cap\overline{\Omega}_2} \varphi(u\nu_2^t + F(u)\nu_2^x)\, ds + \int_{\{x=1\}\cap\overline{\Omega}_3} \varphi(u\nu_3^t + F(u)\nu_3^x)\, ds,$$

where $F(u) = \frac{1}{2}u^2$ and ν_i is the unit outer normal to $\partial\Omega_i$, $i = 1, 2, 3$. Observe that $\{x = t\} \cap \overline{\Omega}_1 = \{x = t\} \cap \overline{\Omega}_2$ and that $\{x = 1\} \cap \overline{\Omega}_2 = \{x = 1\} \cap \overline{\Omega}_3$. We have $\nu_1 = -\nu_2$ along $\{x = t\} \cap \overline{\Omega}_1$ and $\nu_2 = -\nu_3$ along $\{x = 1\} \cap \overline{\Omega}_2$. Because of this and the continuity of u, the first integral cancels the second one and the third integral cancels the fourth one. We conclude that u is a weak solution.

(b) First, notice that $\frac{1-x}{1-t} \geq 0$ in the region satisfying $t < x < 1$ and $t < 1$. But in the same region we also have $x - t < 1 - t$ and $1 - t > 0$, so that

$$0 \leq \frac{1-x}{1-t} \leq \frac{1-t}{1-t} = 1,$$

hence $\frac{1-x}{1-t}$ remains bounded as $t \to 1^-$. In particular, there is no contradiction with question 4 of HW 11. (Technical note: in question 4 of HW 11, we had that u was a classical solution everywhere. Here, the lines $x = t$ and $x = 1$ are excluded, but using the continuity of u across these lines we see that the L^∞ norm is still conserved. In fact, if we use the "correct" definition of L^∞ in measure theoretical terms, then the lines $x = t$ and $x = 1$ could simply be ignored.)

(c) This is like part (a), separating the integral in several regions and using the continuity of u to cancel the boundary terms. The only boundary terms that do not necessarily cancel are along the line $x = \beta t + 1 - \beta$, $t \geq 1$, since v is not continuous across this line. But this is the part done in class, where we showed that the corresponding boundary integrals cancel if and only if $\beta = 1/2$.

Question 2. Show that the function v in the previous problem verifies the Rankine-Hugoniot conditions if an only if $\beta = 1/2$.

Solution 2. Across $x = \beta t + 1 - \beta$, $t \geq 1$, we have $[\![u]\!] = 1$, $[\![F(u)]\!] = 1/2$, and $\sigma = \beta$, hence the result. (Note that we are calling u here what we had called v above.)

Question 3. Formulate the definition of weak solutions, shocks, and the Rankine-Hugoniot conditions for systems of conservation laws. Give a brief sketch of the proof of the Rankine-Hugoniot theorem for systems (you do not have to do all the proof; it suffices to indicate how to modify the $N = 1$ case done in class. Your answer should be two or three sentences long.)

Solution 3. Everything is essentially the same, where now products are replaced by the dot product and equality between vector valued quantities (e.g., the Rankine-Hugoniot conditions in the case of systems) are vector equalities.

Question 4. Show that the $1d$ compressible Euler equations stated in class form a system of conservation laws.

Solution 4. Set $E := \frac{1}{2}v^2 + e$ and $u = (u^1, u^2, u^3) := (\rho, \rho v, \rho E)$. Define, for $u^1 > 0$,

$$F(u) = (F^1(u), F^2(u), F^3(u)) = (F^1(u^1, u^2, u^3), F^2(u^1, u^2, u^3), F^3(u^1, u^2, u^3))$$

by

$$\begin{aligned} F^1(u^1, u^2, u^3) &= u^2, \\ F^2(u^1, u^2, u^3) &= \frac{(u^2)^2}{u^1} + p(u^1, \frac{u^3}{u^1} - \frac{1}{2}\left(\frac{u^2}{u^1}\right)^2), \\ F^3(u^1, u^2, u^3) &= \frac{u^2 u^3}{u^1} + p(u^1, \frac{u^3}{u^1} - \frac{1}{2}\left(\frac{u^2}{u^1}\right)^2)\frac{u^2}{u^1}. \end{aligned}$$

Question 5. Prove that if u is a weak solution that is C^∞ then it is in fact a classical solution.

Solution 5. By assumption we have

$$\int_0^\infty \int_{-\infty}^\infty (\partial_t \varphi u + \partial_x \varphi F(u))\, dx dt + \int_{-\infty}^\infty \varphi(0, x) u(0, x)\, dx = 0.$$

for every test function φ. If the C^∞ function u is not a classical solution, then $\partial_t u + \partial_x(F(u))$ is not zero for some (t_0, x_0). Say that $(\partial_t u + \partial_x(F(u)))|_{(t_0, x_0)} > 0$. Then $\partial_t u + \partial_x(F(u)) > 0$ in a small ball $B_\varepsilon((t_0, x_0))$, and we can assume that $B_\varepsilon((t_0, x_0)) \cap \{t = 0\} = \varnothing$. Take φ to be a test function that is ≥ 0 and whose support lies in $B_\varepsilon((t_0, x_0))$. Integration by parts then gives

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty (\partial_t \varphi u + \partial_x \varphi F(u))\, dx dt + \int_{-\infty}^\infty \varphi(0, x) u(0, x)\, dx &= \int_{B_\varepsilon((t_0, x_0))} (\partial_t \varphi u + \partial_x \varphi F(u))\, dx dt \\ &= -\int_{B_\varepsilon((t_0, x_0))} \varphi(\partial_t u + \partial_x(F(u)))\, dx dt \neq 0, \end{aligned}$$

contradicting the fact the u is a weak solution.