HOMEWORK 11 SOLUTIONS

MATH 3120

Unless stated otherwise, the notation below is as in class.

Question 1. Solve the following problems. In each case, sketch the characteristic curves, and indicate the region in the xy-plane where the solution is defined.

(a) $xu_y - yu_x = u$,

with the condition $u(x, 0) = g(x)$, where g is a given function.

(b) $u_x + u_y = u^2$,

for (x, y) in the region $\{y \geq 0\}$, with the condition $u(x, 0) = g(x)$, where g is a given function. Find the solution in the case $g(x) = x^2$.

(c) $u_x + u_y + u = 1$,

with the condition $u = \sin x$ on $y = x + x^2$, $x > 0$.

Solution 1. (a) Parametrize the initial condition by

$$x_0(s) = s, y_0(s) = 0, u_0(s) = g(s).$$

The characteristic equations are

$$\dot{x} = -y, \tag{1a}$$

$$\dot{y} = x, \tag{1b}$$

$$\dot{u} = u. \tag{1c}$$

Differentiating (1a) with respect to t and using (1b) we find $\ddot{x} + x = 0$, which has solution $x(t, s) = s \cos t$, where we used the initial condition. Similarly we find $y(t, x) = s \sin t$. Equation (1c) can be solved directly and gives, after using the initial condition, $u(t, s) = g(s)e^t$.

Since $y/x = \tan t$ and $x^2 + y^2 = s^2$, we can solve for t and s as functions of x and y, finding

$$u(x, y) = g(\sqrt{x^2 + y^2})e^{\tan^{-1} \frac{y}{x}}.$$

From $x(t, s) = s \cos t$ and $y(t, x) = s \sin t$, we have that the characteristics are circles centered at the origin. The solution is defined for $x > 0$ since we have chosen the positive square root when solving for s. Indeed, notice that

$$\begin{aligned} J = \det \begin{bmatrix} \partial_t x & \partial_s x \\ \partial_t y & \partial_s y \end{bmatrix} &= \partial_t x \partial_s y - \partial_s x \partial_t y \\ &= -s \sin t \sin t - (\cos t) s \cos t = -s, \end{aligned}$$

So that $J(t, 0) = 0$, indicating a potential problem at $s = 0$. (What would happen if we had chosen the negative root?)

(b) We parametrize the initial condition as in (a). The characteristic equations are

$$\dot{x} = 1,$$
$$\dot{y} = 1,$$
$$\dot{u} = u^2.$$

The solutions are $x(t,s) = s + t$, $y(t,s) = t$, and $u(t,s) = \frac{g(s)}{1-tg(s)}$. But $s = x - t = x - y$, hence

$$u(x,y) = \frac{g(x-y)}{1 - yg(x-y)}.$$

The characteristics are straight lines: $y = x - s$. This solution is defined as long as $1 - yg(x-y) \neq 0$. For $g(x) = x^2$, we obtain

$$u(x,y) = \frac{(x-y)^2}{1 - y(x-y)^2}.$$

(c) Parametrize the initial condition as $x_0(s) = s$, $y_0(s) = s + s^2$, $u_0(s) = \sin s$, $s > 0$. The characteristic equations are

$$\dot{x} = 1,$$
$$\dot{y} = 1,$$
$$\dot{u} = 1 - u.$$

We readily find

$$x(t,s) = t + s,\ y(t,s) = t + s + s^2,\ u(t,s) = 1 - (1 - \sin s)e^{-t}.$$

Using the equation for x into the equation for y gives $s = \sqrt{y - x}$, where we chose the positive root according to $x > 0$. Then $t = x - \sqrt{y-x}$, thus

$$u(t,x) = 1 - (1 - \sin\sqrt{y-x})e^{-x+\sqrt{y-x}}.$$

The solution is defined in the region

$$\{(x,y)\,|\,0 < x < y\}$$

The characteristic curves are lines $y = x + s^2$. Notice that the derivatives of u are not defined at $(0,0)$. Computing the Jacobian, we find

$$J(0,s) = 2s,$$

and we see that the transversality conditions fails at $s = 0$ (which corresponds to $(0,0)$). The geometric interpretation of this, discussed in class, can be easily seen here. The characteristic curve for $s = 0$, $y = x$, is tangent to $\Gamma(s)$ at $s = 0$. Thus, the theorem of existence and uniqueness of solutions does not guarantee a solution valid for $x = y = 0$.

Question 2. Solve

$$uu_x - uu_y = u^2 + (x+y)^2,$$

with initial condition $u(x,0) = 1$.

Hint: after writing the characteristic equations, identify an equation satisfied by $x + y$.

Solution 2. Parametrize the initial condition by $x_0(s) = s$, $y_0(s) = 0$, $u_0(s) = -1$. The characteristic equations are

$$\dot{x} = u, \tag{2a}$$

$$\dot{y} = -u, \tag{2b}$$

$$\dot{u} = u^2 + (x+y)^2, \tag{2c}$$

Adding (2a) and (2b), we obtain

$$\partial_t(x+y) = 0,$$

which, in light of the initial condition, gives

$$x + y = s. \tag{3}$$

Using (3) into (2c) produces $\dot{u} = u^2 + s^2$, which can be integrated to

$$\frac{1}{s}\tan^{-1}\left(\frac{u}{s}\right) = t + g(s).$$

Using the initial condition we find $g(s) = \frac{1}{s}\tan^{-1}\left(\frac{1}{s}\right)$, thus

$$u(t,s) = s\tan\left(st + \tan^{-1}\left(\frac{1}{s}\right)\right). \tag{4}$$

Using (4) into (2a) gives

$$\dot{x} = s\tan\left(st + \tan^{-1}\left(\frac{1}{s}\right)\right).$$

Integrating with respect to t and using the initial condition,

$$x(t,s) = -\ln\left|\frac{\cos(st + \tan^{-1}\left(\frac{1}{s}\right))}{\cos\tan^{-1}\left(\frac{1}{s}\right)}\right| + s. \tag{5}$$

Using (3) into the last term of (5) gives

$$y(t,s) = \ln\left|\frac{\cos(st + \tan^{-1}\left(\frac{1}{s}\right))}{\cos\tan^{-1}\left(\frac{1}{s}\right)}\right|. \tag{6}$$

From (6) we get

$$st = \cos^{-1}\left(\frac{se^y}{\sqrt{1+s^2}}\right) - \tan^{-1}\left(\frac{1}{s}\right), \tag{7}$$

where we used the identity $\cos\tan^{-1} z = \frac{1}{\sqrt{1+z^2}}$. Using (7) so replace st and (3) to replace s in (4) finally gives

$$u(x,y) = e^{-y}\sqrt{1 + (x+y)^2 - (x+y)^2 e^{2y}},$$

where we used the identity $\tan\cos^{-1} z = \frac{\sqrt{1-z^2}}{z}$.

Question 3. In this question, you will provide a blow-up proof for Burgers' equation different than the one given in class.

(a) Differentiate the equation and show that the variable $\psi := \partial_x u$ satisfies the equation

$$\partial_t\psi + u\partial_x\psi = -\psi^2. \tag{8}$$

(b) Show that (8) implies that $y(t) := \psi(t, x(t, \alpha))$ satisfies the Riccati equation $\dot{y} = -y^2$ along the characteristics $(t, x(t, \alpha))$.

(c) Use your knowledge of ODE to conclude, from the Riccati equation, blow-up for Burgers.

Solution 3. (a) This is a straightforward computation.

(b) Using the chain rule and the characteristic equations,

$$\begin{aligned}\frac{dy(t)}{dt} &= \partial_t \psi(t, x(t, \alpha)) + \frac{dx(t, \alpha)}{dt} \partial_x \psi(t, x(t, \alpha)) \\ &= \partial_t \psi(t, x(t, \alpha)) + u(t, x(t, \alpha)) \partial_x \psi(t, x(t, \alpha)) \\ &= -(\psi(t, x(t, \alpha)))^2 \\ &= -(y(t))^2.\end{aligned}$$

(c) The solution to the Riccati equation is

$$y(t) = \frac{1}{1/y(0) + t},$$

leading to blow-up if $y(0) = \partial_x u(0, x) < 0$ for some x.

Question 4. Define

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} := \sup_{x \in \mathbb{R}} |u(t, x)|.$$

Write Burgers' equation as an ODE along the characteristics (similarly to what you did for ψ in the previous problem) to conclude that

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} = \|u(0, \cdot)\|_{L^\infty(\mathbb{R})} = \|h\|_{L^\infty(\mathbb{R})},$$

i.e., the L^∞ norm is conserved over time.

Solution 4. We have

$$\begin{aligned}\frac{d}{dt} u(t, x(t, \alpha)) &= \partial_t u(t, x(t, \alpha)) + \frac{dx(t, \alpha)}{dt} \partial_x u(t, x(t, \alpha)) \\ &= \partial_t u(t, x(t, \alpha)) + u(t, x(t, \alpha)) \partial_x u(t, x(t, \alpha)) \\ &= 0.\end{aligned}$$

Integrating with respect to t gives

$$u(t, x(t, \alpha)) = u(0, x(0, \alpha)) = u(0, \alpha).$$

As long as the characteristics do not intersect, for each t_0 there exists a one-to-one correspondence between $\{t = t_0\}$ and $\{t = 0\}$. Hence, taking the sup over α produces the result.

Question 5. Consider the eikonal equation:

$$u_x^2 + u_y^2 = n^2, \tag{9}$$

where $n = n(x, y)$ is a given function. The eikonal equation has important applications in optics.

The goal of this problem is to show how the method of characteristics can be used to solve the eikonal equation, which is a fully non-linear first order PDE.

Assume that an initial condition for (9) is given in the form of a parametrized curve $\Gamma(s) = (x_0(s), y_0(s), u_0(s))$.

(a) Show that (9) is equivalent to $(u_x, u_y, n^2)\cdot(u_x, u_y, -1) = 0$ and interpret this geometrically.

(b) Using (a), explain why it makes sense to consider the following system of characteristic equations for $x = x(t,s)$, $y = y(t,s)$, and $u = u(t,s)$ (recall the geometric meaning of the characteristic curves)

$$\dot{x} = u_x \tag{10a}$$
$$\dot{y} = u_y \tag{10b}$$
$$\dot{u} = n^2 \tag{10c}$$

(c) From equations (10) and (9), derive

$$\ddot{x} = \frac{1}{2}\partial_x n^2 \tag{11a}$$
$$\ddot{y} = \frac{1}{2}\partial_y n^2 \tag{11b}$$
$$\dot{u} = n^2 \tag{11c}$$

(d) Show that the solution to (9) is given by

$$u(x(t,s), y(t,s)) = u(x_0(s), y_0(s)) + \int_0^t (n(x(\tau,s), y(\tau,s)))^2 \, d\tau,$$

where $(x(\tau,s), y(\tau,s))$ is a solution to (2a)-(2b).

Solution 5. Computing $(u_x, u_y, n^2)\cdot(u_x, u_y, -1)$ we see that (9) is equivalent to $(u_x, u_y, n^2)\cdot(u_x, u_y, -1) = 0$. Since $(u_x, u_y, -1)$ is normal to the graph of u, we see that (u_x, u_y, n^2) must be tangent to it. As the characteristic equations correspond to equations for curves lying on the graph of u, we see that we should consider (10).

Using the chain rule and equation (10a), we find

$$\begin{aligned}\ddot{x} = \frac{d}{dt}\dot{x} = u_{xx}\frac{dx}{dt} + u_{xy}\frac{dy}{dt} \\ &= u_{xx}u_x + u_{xy}u_y = \frac{1}{2}\partial_x(u_x^2 + u_y^2) \\ &= \frac{1}{2}\partial_x n^2,\end{aligned}$$

which is (11a). Similarly we obtain (11b).

Finally, from our definitions and the chain rule we have that

$$\begin{aligned}\frac{du}{dt} &= u_x\frac{dx}{dt} + u_y\frac{dy}{dt} \\ &= u_x^2 + u_y^2 \\ &= n^2.\end{aligned}$$

Integrating in t yields the final answer.

Question 6. Solve (9) when $n(x,y) = 1$ and with initial condition $u = 1$ on the curve $y = 2x$.

Solution 6. Parametrize the initial condition as

$$x_0(s) = s, y_0(s) = 2s, u_0(s) = 1.$$

Since (11a) and (11b) are second order ODEs, we need initial conditions for $\dot{x}$ and $\dot{y}$ as well, which we denote $\dot{x}_0(s)$ and $\dot{y}_0(s)$. From (10a), (10b), and (9) we know that

$$(\dot{x}_0)^2 + (\dot{y}_0)^2 = n^2 = 1. \tag{12}$$

Differentiating $u_0(s) = 1$ with respect to s and using (10a)-(10b) produces

$$\dot{x}_0 + 2\dot{y}_0 = 0. \tag{13}$$

Solving (12)-(13) yields

$$\dot{x}_0 = \frac{2}{\sqrt{5}}, \dot{y}_0 = -\frac{1}{\sqrt{5}}.$$

We can now solve (11a)-(11b) with the above initial conditions to find

$$x(t,s) = \frac{2}{\sqrt{5}}t + s, y(t,s) = -\frac{1}{\sqrt{5}}t + 2s.$$

Using these expressions in the formula for u gives

$$u(t,s) = t + 1.$$

We can solve for (t,s) in terms of (x,y) to finally obtain

$$u(x,y) = 1 + \frac{2x - y}{\sqrt{5}}.$$