HOMEWORK 10 SOLUTIONS

MATH 3120

Unless stated otherwise, the notation below is as in class.

Question 1. Consider continuous dependence on the data for the wave equation in $3d$, where smallness on the data part is measured with respect to the norm

$$\|f\|_2 := \int_{\mathbb{R}^3} (|D^2 f(y)| + |Df(y)| + |f(y)|)\, dy.$$

Give a precise formulation of the continuous dependence on the data and prove your statement.

Hint: Use the estimate of Question 3 in HW 6 as a basis for your statement, and give a similar proof (now you have to also account for $t < 1$).

Solution 1. We formulate it as follows. Let (u_0, u_1) and (v_0, v_1) be two data sets for the wave equation, and let u and v be the respective solutions. Solutions depend continuously on the data if given $\varepsilon > 0$ and $t > 0$, there exists a $\delta > 0$ such that if

$$\|u_0 - v_0\|_2 + \|u_1 - v_1\|_2 < \delta,$$

then

$$|u(t,x) - v(t,x)| < \varepsilon$$

for all $x \in \mathbb{R}^3$.

We now prove the statement. Set $w_0 = u_0 - v_0$, $w_1 = u_1 - v_1$, and $w = u - v$. By Kirchhoff's formula:

$$w(t,x) = \frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} (w_0(y) + t w_1(y) + \nabla w_0(y) \cdot (y - x))\, dS(y).$$

Proceeding as in HW 6, we find

$$\left| \frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} w_0(y)\, dS(y) \right| \le \frac{C}{t^2} \int_{\mathbb{R}^3} (|\nabla w_0(y)| + |w_0(y)|)\, dy,$$

$$\left| \frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} w_1(y)\, dS(y) \right| \le \frac{C}{t} \int_{\mathbb{R}^3} (|\nabla w_1(y)| + |w_1(y)|)\, dy,$$

and

$$\left| \frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} \nabla w_0(y) \cdot (y - x)\, dS(y) \right| \le \frac{C}{t} \int_{\mathbb{R}^3} |D^2 w_0(y)|\, dy.$$

Combining the above we find

$$|w(t,x)| \le C \max\{\frac{1}{t}, \frac{1}{t^2}\}(\|w_0\|_2 + \|w_1\|_2),$$

which implies the result.

Question 2. In the equations below, identify the functions $a(t,x,u)$, $b(t,x,u)$, and $c(t,x,u)$ and write the corresponding characteristic system.

(a) $(1+t^2)\partial_t u + 3\partial_x u + u^2 = 0.$

(b) $\sin(x)e^t u_t + |u|^3 u_x = 0.$

Solution 2. The characteristic system is always $\dot{t} = a$, $\dot{x} = b$, $\dot{u} = -c$. We have: (a) $a(t,x,u) = (1+t^2)$, $b(t,x,u) = 3$, $c(t,x,u) = u^2$. (b) $a(t,x,u) = \sin(x)e^t$, $b(t,x,u) = |u|^3$, $c(t,x,u) = 0$.

Question 3. Solve the problem below using the method of characteristics and give a description of the (projected) characteristics.

$$x\partial_t u - t\partial_x u - u = 0,$$
$$u(0,x) = h(x).$$

Solution 3. The characteristic system reads

$$\dot{t} = x,\ \dot{x} = -t, \dot{u} = 1.$$

From the third equation and the initial condition we find $u(\tau,\alpha) = e^\tau h(\alpha)$. Differentiating $\dot{t}$ and using the $\dot{x}$ equation we find $\ddot{t} + t = 0$. Using the initial condition and that $\dot{t} = x$ also holds for $\tau = 0$ we find $t(\tau,\alpha) = \alpha\sin(\tau)$. Similarly we find $x(\tau,\alpha) = \alpha\cos(\tau)$. Inverting these relations we find

$$u(t,x) = h(\sqrt{t^2+x^2})e^{\arctan(t/x)}.$$

The characteristics are circles about the origin.

Question 4. Does the transversality condition hold for the problem of question 3? What can you say about uniqueness and how is it related to the solution you found?

Solution 4. We find $J = x$, so the transversality condition fails at $x = 0$. In our solution above, we chose the positive sign when we took the square root, restricting thus the solution to the region $x > 0$.