# MATH 3120

VANDERBILT UNIVERSITY

## Contents

## 1. Abbreviations

- ODE = ordinary differential equation
- PDE = partial differential equation
- LHS = left hand side
- RHS = right hand side
- w.r.t = with respect to
- $\Rightarrow$ = implies
- LHS := RHS means that the LHS is defined by the RHS.
- nd (e.g. 1d, 2d, ...) = n dimensional

## 2. What are partial differential equations and why do we study them

Recall that an ordinary differential equation (ODE) is an equation involving an unknown function of a single variable and some of its derivatives. For example,

$$\frac{dy}{dx} + y^2 = 0, \quad \text{(unknown } y\text{, non-linear 1st order)} \tag{2.1}$$

$$y'' + y' + y = 0, \quad \text{(unknown } y\text{, linear 2nd order)} \tag{2.2}$$

$$(x^2 - 1)\frac{d^2u}{d^2x} + u = 0, \quad \text{(unknown } u\text{, linear 2nd order)} \tag{2.3}$$

$$\tag{2.4}$$

are ODEs. We can also have systems of ODEs, i.e., a system of equations involving two or more unknown functions of a single variable and their derivatives. For example,

$$\begin{cases} \frac{dy}{dt} + x = 0 \quad \text{(unknowns: } y \text{ and } x\text{, linear, 1st order)} \\ \frac{dx}{dt} - y = 0 \end{cases} \tag{2.5}$$

$$\begin{cases} u'' + v^2 = 0 \quad \text{(unknowns: } u\text{, } v\text{, } w\text{, non linear)} \\ v'' + v^2 = 0 \\ w'' + w' + wv = 0 \end{cases} \tag{2.6}$$

are systems of ODEs. As we learn in ODE courses, one typically studies ODEs because many phenomena in science and engineering are modeled with ODEs. A limitation of ODEs, however, is that they are restricted to functions of a single variable, whereas many important phenomena are described by functions of several variables . For instance, suppose we want to describe the temperature $T$ in a room. It will in general be different at different positions in the room, so $T$ is a function of $(x, y, z)$. $T$ can also change over time, thus $T = T(t, x, y, z)$. An equation involving $T$ and its derivatives can then have derivatives with respect to any of the variables $t$, $x$, $y$, or $z$, which will be partial derivatives, $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$. This will be a partial differential equation. Formally:

**Definition 2.1.** A partial differential equation is an equation involving an unknown function of two or more variables and some of its (partial) derivatives. A system of PDEs is a system of equations involving two or more unknown functions of two or more variables and some of their (partial) derivatives. A solution to a PDE (or system) is a function that verifies the PDE.

**Notation 2.2.** Since most of the time we sill be dealing with functions of several variables, the derivatives will be partial derivatives, but we will often omit the word "partial", referring simply to "derivatives." We will also often omit "system", and use PDE to refer to both a single equation and systems of PDEs.

Besides applications to science and engineering PDEs are also used in many branches of mathematics, such as in complex analsis or geometry (see in particular Ricci flow and the Poincaré conjecture). PDEs are also studied in mathematics for their own sake, i.e., from a "pure" point of view.

## 3. Examples and notation

We will now give examples of PDEs. Along the way, we will introduce some notation that will be used throughout.

**Remark 3.1.** As it was the case for ODEs, when we introduce a PDE, strictly speaking we have to specify where the equation is defined. We will often ignore this for the time being until we get to some more formal aspects of PDE theory.

### 3.1. **Laplace's equation:**

$$\triangle u = 0,$$

where $\triangle$ is the Laplacian operator defined by

$$\triangle := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

so explicitly Laplace's equation reads:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

We will often denote coordinates in $\mathbb{R}^3$ by $(x^1, x^2, x^3)$, in which case we write $\triangle$ as $\triangle = \frac{\partial^2}{\partial (x^1)^2} + \frac{\partial^2}{\partial (x^2)^2} + \frac{\partial^2}{\partial (x^3)^2}$. We write expressions of the form $u = u(x^1, x^2, x^3)$ to indicate the variables that a function depends on, e.g., in this case that $u$ is a function of $x^1$, $x^2$, and $x^3$. We can also consider Laplace's equations for a function of $x^1, x^2, \cdots, x^n$, for some arbitrary $n$, $u = u(x^1, x^2, \cdots, x^n)$, in which case

$$\triangle := \frac{\partial^2}{\partial (x^1)^2} + \frac{\partial^2}{\partial (x^2)^2} + \cdots + \frac{\partial^2}{\partial (x^n)^2},$$

so Laplace's equation reads

$$\triangle u = \frac{\partial^2 u}{\partial (x^1)^2} + \frac{\partial^2 u}{\partial (x^2)^2} + \cdots + \frac{\partial^2 u}{\partial (x^n)^2} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial (x^i)^2} = 0$$

Laplace's equation has many applications. Typically, $u$ represents the density of some quantity (e.g., a chemical concentration). Closely related to Laplace's equation is the Poisson equation:

$$\triangle u = f,$$

where $f$ is a given function.

3.2. **Heat equation or diffusion equation.**

$$\partial_t u - \triangle u = 0.$$

This equation has many applications. For example, $u$ can represent the temperature so $u(t, x^1, x^2, x^3)$ is the temperature at the point $(x^1, x^2, x^3)$ at instant $t$. More generally $u$ can represent the concentration of some quantity that diffuses over time.

**Notation 3.2.** Throughout these notes, we will use $t$ to denote a time variable, unless otherwise specified.

**Remark 3.3.** The heat equation is also written as $\partial_t u - k\triangle u = 0$, where $k$ is a constant known as diffusivity. In most of these notes, we will ignore physical constants in the equations, setting them equal to 1.

3.3. **Wave equation.**

$$u_{tt} - \triangle u = 0$$

(Here we recall the notation $u_t = \partial_t u = \frac{\partial u}{\partial t}$, $u_{tt} = \partial_{tt}^2 u = \frac{\partial^2 u}{\partial t^2}$ etc.). This equation describes a wave propagating in a medium (e.g., a radio wave propagating in space); $u$ is the amplitude of the wave.

Sometimes one writes $u_{tt} - c^2\triangle u = 0$ where the constant $c$ is the speed of propagation of the wave (we will see later on why $c$ is indeed the speed of propagation).

3.4. **Schödinger equation.**

$$i\frac{\partial \Psi}{\partial t} + \triangle \Psi + V\Psi = 0,$$

where $i$ is the complex unit $i^2 = -1$, $V = V(t, x^1, x^2, x^3)$ is a known function called the potential (whose specific form depends on the problem we are studying), and the unknown function $\Psi$, called the wave-function, is a complex function, i.e.,

$$\Psi = u + iv,$$

where $u$ and $v$ are real valued functions.

The Schrödinger equation is the fundamental equation of quantum mechanics.

3.5. **Burger's equation.**

$$u_t + uu_x = 0.$$

Burger's equation has applications in the study of shock waves.

3.6. **Maxwell's equations.**

$$\begin{cases} \partial_t E - \operatorname{curl} B = -J, \\ \partial_t B + \operatorname{curl} E = 0, \\ \operatorname{div} E = \rho, \\ \operatorname{div} B = 0, \end{cases} \tag{3.1}$$

where the $E$ and $B$ are vector fields that are the unknown functions (or vector valued functions), so they have three components each:

$$E = (E^1, E^2, E^3), \tag{3.2}$$

$$B = (B^1, B^2, B^3), \tag{3.3}$$

div and curl are the divergence and curl operators, sometimes written as $\nabla\cdot$ and $\nabla\times$, respectively (curl is also called the rotational). Let us recall the definition of these operators: for any vector field $X = (X^1, X^2, X^3)$, we have

$$\operatorname{div} X := \partial_1 X^1 + \partial_2 X^2 + \partial_3 X^3,$$

and

$$\operatorname{curl} X := (\partial_2 X^3 - \partial_3 X^2, -\partial_1 X^3 + \partial_3 X^1, \partial_2 X^3 - \partial_3 X^2$$

where we have introduced the following notation:

$$\partial_i := \frac{\partial}{\partial x^i}.$$

$E$ and $B$ represent the electric and magnetic fields respectively. $\rho$ represents the charge density and $J$ the current density, which are given.

Maxwell's equations are the fundamental equations of electromagnetism.

**Notation 3.4.** Note that the above, we did not denote vectors with an "arrow" i.e., $\overrightarrow{E}$ and $\overrightarrow{B}$, as usually done in calculus. We will avoid using arrows for vectors - it will always be clear from the context if a quantity is a scalar, a vector field, etc. We also denote the components or entries of a vector with superscripts and not with subscripts as usually in calculus (i.e., $X^i$ and not $X_i$, but see below for exceptions).

Similarly, we will denote points in space by a single letter without an arrow, e.g., $x = (x^1, x^2, x^3)$ in $\mathbb{R}^3$, or more generally $x = (x^1, x^2, x^3, \cdots, x^4)$ in $\mathbb{R}^4$. So, sometimes we write expressions like $u = u(t, x)$ instead of $u = u(t, x^1, x^2, x^3)$.

**Notation 3.5.** The curl can be written in a compact form as

$$\underbrace{(\operatorname{curl} X)^i}_{\text{meaning the } i^{th} \text{ component of the vector curl } X} = \epsilon^{ijk}\partial_j X_k.$$

In this expression, the following convention is adopted. $\epsilon$ is the totally anti-symmetric symbol, defined as

$$\epsilon^{ijk} := \begin{cases} +1 & \text{if } ijk \text{ is an even permutation of } 123 \\ -1 & \text{if } ijk \text{ is an odd permutation of } 123 \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

E.g., $\epsilon^{123} = 1$, $\epsilon^{231} = 1$, $\epsilon^{213} = -1$, $\epsilon^{112} = 0$. $X_k$ means $X^k$, but we write it here with a subscript because of the following summation convention which will be used throughout:

**Notation 3.6.** When an index (such as $i$, $j$, etc.) appears repeated in an expression, once upstairs and once downstairs, it is summed over its range.

E.g., we can write the divergence as

$$\operatorname{div} X = \partial_i X^i = \sum_{i=1}^{3} \partial_i X^i = \partial_1 X^1 + \partial_2 X^2 + \partial_3 X^3.$$

**Remark 3.7.** We will give another interpretation to $X_k$ (i.e., $X^k$ but with the index downstairs) which will make our conventions more systematic, later on.

In the expression for curl, for example:

$$(\operatorname{curl} X)^2 = \epsilon^{2jk}\partial_j X_k \tag{3.5}$$
$$= \epsilon^{213}\partial_1 X_3 + \epsilon^{231}\partial_3 X_1 \tag{3.6}$$
$$= \partial_1 X_3 + \partial_3 X_1. \tag{3.7}$$

We also sometimes use the notation

$$\operatorname{curl}^i X = (\operatorname{curl} X)^i.$$

3.7. **Euler and Navier-Stokes equations.**

$$\begin{cases} \partial_t \rho + (u \cdot \nabla)\rho + \rho \operatorname{div} u = 0 \\ \rho(\partial_t u + (u \cdot \nabla)u) + \nabla p = \mu \triangle u \end{cases} \tag{3.8}$$

These equations describe the motion of a fluid. The first equation is sometimes called the continuity equations (conservation of mass) and the second one the momentum equation (conservation of momentum)

$\rho = \rho(t, x)$ is a scalar function representing the fluid's density and $u = u(t, x)$ is a vector field representing the fluid's velocity. $\rho$ and $u$ are the unknowns. $p$ is a given function of $\rho$, i.e., $p = p(\rho)$ (e.g., $p(\rho) = \rho^2$). $p$ represents the pressure of the fluid. $\mu \geq 0$ is a constant known as the viscosity of the fluid. $\nabla$ is the gradient operator; recall that $\nabla f := (\partial_1 f, \partial_2 f, \partial_3 f)$, where $f$ is a scalar function, so the $i^{th}$ component reads $(\nabla f)^i = \partial_i f$; we also write $\nabla^i f$ for $(\nabla f)^i$.

$u \cdot \nabla$ is the operator

$$u \cdot \nabla = u^i \partial_i \tag{3.9}$$
$$= u^1\partial_1 + u^2\partial_2 + u^3\partial_3. \tag{3.10}$$

When $u \cdot \nabla$ acts on a vector field it does so componentwise. $\triangle$ also acts on a vector field componentwise.

These equations are known as the Navier-Stokes equations if $\mu > 0$ and Euler equations if $\mu = 0$. They are the fundamental equations of hydrodynamics.

In models where the density is assumed to be constant, we take $\rho = 1$, we have the incompressible Euler or Navier-Stokes equations:

$$\begin{cases} \operatorname{div} u = 0 \\ \partial_t u + (u \cdot \nabla)u + \nabla p = \mu \triangle u \end{cases} \tag{3.11}$$

In this case, however, it is no longer assumed that $p = p(\rho)$, and $p$ is given by some other expression (we will see this later).

3.8. **Other examples.** There are many other important PDEs that we will not have time to discuss. We mention a few more of them, without writing them explicitly:
Einstein's equations: fundamental equations of general relativity
Yang-Mills equations: fundamental equations of quantum field theory
Black-Scholes equation: models the price of European options.

**Remark 3.8.** The concepts of the order of a PDE and of homogeneous vs. non-homogeneous PDEs are defined similarly to their analogs in ODEs. We will define linear and non-linear PDEs later on, but this definition is also similar to ODEs and readers should be able to identify which of the above examples are linear or non-linear PDEs.

3.9. **Theory and Example.** Before investigating more general and theoretical aspects of PDEs, it is useful to first consider a few specific equations that can be solved explicitly. Thus, the beginning will be more computational and equation specific. Later on we will consider more robust aspects of the general theory of PDEs.

## 4. Schrödinger equation

4.1. **Introduction.** Our goal is to investigate solutions to the *Schrödinger equation*,

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2\mu}\Delta\Psi + V\Psi, \tag{4.1}$$

where $i$ is the imaginary number $i^2 = -1$; $\hbar = 1.51 \times 10^{-27}\, erg\, s$ is Planck's constant; $\mu$ is a positive constant called the mass; $V = V(t,x) : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$ is called the potential function; and the unknown is the complex-valued function $\Psi = \Psi(t,x) : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}$ called the wave-function. The variables $t$ and $x$ represent, respectively, the time and space variables.

The Schrödinger equation describes the dynamics of a particle of mass $\mu$ interacting with a potential $V$, according to the laws of Quantum Mechanics. The physical interpretation of $\Psi$ is as follows. If $U \subseteq \mathbb{R}^3$, then

$$\int_U |\Psi(t,x)|^2\, dx$$

represents the probability of finding the particle in the region $U$ at a time $t$. In particular, one must have

$$\int_{\mathbb{R}^3} |\Psi(t,x)|^2\, dx = 1. \tag{4.2}$$

Notice that, upon multiplying $\Psi$ by a suitable constant, condition (4.2) can always be fulfilled as long as

$$\int_{\mathbb{R}^3} |\Psi(t,x)|^2\, dx < \infty. \tag{4.3}$$

Our treatment will be based on [2–4], to which the student is referred for more details.

4.2. **Separation of variables for a time-independent potential.** We shall assume that $V$ does not depend on time, i.e., $V(t,x) = V(x)$. We will have to divide several expressions by $\Psi$. In order to make this sensible, it will be assumed that $\Psi$ does not vanish (or, at least, does not vanish on an open set). Look for solutions of the form

$$\Psi(t,x) = T(t)\psi(x), \tag{4.4}$$

Plugging (4.4) into (4.1) gives

$$i\hbar\frac{T'}{T} = -\frac{\hbar^2}{2\mu}\frac{1}{\psi}\Delta\psi + V,$$

The left-hand side depends only on $t$, whereas the right-hand side depends only on $x$. Thus, both sides have to be equal to a constant, which we denote be $E$. Therefore

$$i\hbar T' = ET, \tag{4.5}$$

and

$$-\frac{\hbar^2}{2\mu}\Delta\psi + V\psi = E\psi. \tag{4.6}$$

Equation (4.5) is easily solved. Its solution is

$$T(t) = e^{-\frac{iE}{\hbar}t}, \tag{4.7}$$

where we ignored an arbitrary constant of integration (such constants will be neglected throughout, as an overall constant of integration can be fixed at the very end via condition (4.2)).

4.2.1. *The time-independent Schrödinger equation.* We now focus on (4.6), known as the *time-independent Schrödinger equation.* To solve it, we assume further that $V$ is *radially symmetric*, i.e., that $V(x) = V(\sqrt{x_1^2 + x_2^2 + x_3^2})$ or, in spherical coordinates, that $V = V(r)$. This assumption suffices to treat many physical systems of interest.

Recall the expression for the Laplacian in spherical coordinates,

$$\Delta = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta_{S^2}, \tag{4.8}$$

where

$$\Delta_{S^2} = \partial_\phi^2 + \frac{\cos\phi}{\sin\phi}\partial_\phi + \frac{1}{\sin^2\phi}\partial_\theta^2 \tag{4.9}$$

is the *Laplacian on the unit sphere*, $r \in [0, \infty)$, $\phi \in [0, \pi]$, and $\theta \in [0, 2\pi)$. From now on we shall work in spherical coordinates.

We suppose that

$$\psi(r, \phi, \theta) = R(r)Y(\phi, \theta). \tag{4.10}$$

Plugging (4.10) into (4.6), and using (4.8),

$$-\frac{\hbar^2}{2\mu}\frac{r^2}{R}\left(R'' + \frac{2}{r}R'\right) + (V - E)r^2 = \frac{\hbar^2}{2\mu}\frac{1}{Y}\Delta_{S^2}Y.$$

Since the left-hand side depends only on $r$ and the right-hand side only on $(\phi, \theta)$, both sides must be equal to a constant, which we denote by $-a$. Thus,

$$-\frac{\hbar^2}{2\mu}\left(R'' + \frac{2}{r}R'\right) + \left(V + \frac{a}{r^2}\right)R = ER, \tag{4.11}$$

and

$$\frac{\hbar^2}{2\mu}\Delta_{S^2}Y = -aY. \tag{4.12}$$

4.2.2. *The angular equation.* We first investigate (4.12), which, in light of (4.9), becomes

$$\partial_\phi^2 Y + \frac{\cos\phi}{\sin\phi}\partial_\phi Y + \frac{1}{\sin^2\phi}\partial_\theta^2 Y = -\frac{2a\mu}{\hbar^2}Y.$$

Supposing

$$Y(\phi,\theta) = \Phi(\phi)\Theta(\theta), \tag{4.13}$$

one gets

$$-\frac{\Theta''}{\Theta} = \frac{\sin^2\phi}{\Phi}\Phi'' + \frac{\sin\phi\cos\phi}{\Phi}\Phi' + \frac{2a\mu\sin^2\phi}{\hbar^2}. \tag{4.14}$$

Once more, both sides ought to be equal to a constant, which we denote by $b$. One equation becomes

$$\Theta'' = -b\Theta. \tag{4.15}$$

To solve (4.15), we need to analyze the cases $b > 0$, $b = 0$, and $b < 0$. Notice the following boundary condition: the points with coordinates $\theta$ and $\theta + 2\pi$ must be identified as they correspond to the same point in $\mathbb{R}^3$. Thus,

$$\Theta(\theta + 2\pi) = \Theta(\theta). \tag{4.16}$$

We immediately see that the case $b < 0$ does not yield a solution satisfying (4.16); $b = 0$ and (4.16) give $\Theta$ =constant; and $b > 0$ along with (4.16) give that $\Theta$ is a linear combination of $\cos(\sqrt{b}\theta)$ and $\sin(\sqrt{b}\theta)$. Moreover, $2\pi$-periodicity requires that $\sqrt{b} = integer$. All these cases can be summarized by setting

$$b = m^2,\ m \in \mathbb{Z}, \tag{4.17}$$

and writing

$$\Theta(\theta) = e^{im\theta}. \tag{4.18}$$

Next, we move to the $\Phi$-equation. From (4.14) and (4.17), one has

$$\frac{\sin\phi}{\Phi}\frac{d}{d\phi}\left(\sin\phi\frac{d\Phi}{d\phi}\right) - m^2 = -\lambda\sin^2\phi, \tag{4.19}$$

where

$$\lambda = \frac{2\mu}{\hbar^2}a, \tag{4.20}$$

and we used the product rule to rewrite the terms involving derivatives. In order to solve (4.19), let us make the following change of variables,

$$x = \cos\phi,\ 0 \le \phi \le \pi.$$

Notice that this change of variables is well-defined since cos is one-to-one for $0 \le \phi \le \pi$. The chain rule gives

$$\sin\phi\frac{d}{d\phi} = \sin\phi\frac{dx}{d\phi}\frac{d}{dx} = -\sin^2\phi\frac{d}{dx} = (\cos^2\phi - 1)\frac{d}{dx} = (x^2-1)\frac{d}{dx},$$

so that (4.19) becomes

$$\frac{d}{dx}\left((1-x^2)\frac{d\Phi}{dx}\right) + \left(\lambda - \frac{m^2}{1-x^2}\right)\Phi = 0. \tag{4.21}$$

To solve (4.21), we seek for a solution of the form

$$\Phi(x) = (1-x^2)^{\frac{|m|}{2}} \frac{d^{|m|}P(x)}{dx^{|m|}}, \tag{4.22}$$

where $P$ solves

$$(1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \lambda P = 0. \tag{4.23}$$

To see that this works, differentiate (4.23) $|m|$ times, obtaining

$$(1-x^2)\frac{d^{|m|+2}P}{dx^{|m|+2}} - 2(|m|+1)x\frac{d^{|m|+1}P}{dx^{|m|+1}} + (\lambda - |m|(|m|+1))\frac{d^{|m|}P}{dx^{|m|}} = 0. \tag{4.24}$$

Students are encouraged to verify (4.24) (compute the first few derivatives to see that a pattern as (4.24) emerges). On the other hand, let $\widetilde{\Phi}$ be defined by

$$\Phi(x) = (1-x^2)^{\frac{|m|}{2}}\widetilde{\Phi}(x) \tag{4.25}$$

and plug this into (4.21). Computing the derivative terms,

$$\begin{aligned}
\frac{d}{dx}\left((1-x^2)\frac{d}{dx}\left((1-x^2)^{\frac{|m|}{2}}\widetilde{\Phi}\right)\right) &= \frac{d}{dx}\left(\frac{|m|}{2}(-2x)(1-x^2)^{\frac{|m|}{2}}\widetilde{\Phi} + (1-x^2)^{\frac{|m|}{2}+1}\frac{d\widetilde{\Phi}}{dx}\right) \\
&= (1-x^2)^{\frac{|m|}{2}+1}\frac{d^2\widetilde{\Phi}}{dx^2} + (1-x^2)^{\frac{|m|}{2}}\frac{d\Phi}{dx}\left(\left(\frac{|m|}{2}+1\right)(-2x) + \frac{|m|}{2}(-2x)\right) \\
&+ \frac{|m|}{2}\left((-2x)\frac{|m|}{2}(1-x^2)^{\frac{|m|}{2}-1}(-2x) - 2(1-x^2)^{\frac{|m|}{2}}\right)\widetilde{\Phi} \\
&= (1-x^2)^{\frac{|m|}{2}+1}\frac{d^2\widetilde{\Phi}}{dx^2} - 2x(1-x^2)^{\frac{|m|}{2}}(|m|+1)\frac{d\widetilde{\Phi}}{dx} + \frac{|m|}{2}(1-x^2)^{\frac{|m|}{2}}\left(\frac{2|m|x^2}{1-x^2} - 2\right)\widetilde{\Phi} \\
&= (1-x^2)^{\frac{|m|}{2}}\left((1-x^2)\frac{d^2\widetilde{\Phi}}{dx^2} - 2x(|m|+1)\frac{d\widetilde{\Phi}}{dx} + |m|\left(\frac{|m|x^2}{1-x^2} - 1\right)\widetilde{\Phi}\right).
\end{aligned}$$

By (4.21), this has to equal

$$-\left(\lambda - \frac{m^2}{1-x^2}\right)\Phi = -\left(\lambda - \frac{m^2}{1-x^2}\right)(1-x^2)^{\frac{|m|}{2}}\widetilde{\Phi}(x),$$

what gives, after canceling $(1-x^2)^{\frac{|m|}{2}}$,

$$(1-x^2)\frac{d^2\widetilde{\Phi}}{dx^2} - 2x(|m|+1)\frac{d\widetilde{\Phi}}{dx} + \lambda\widetilde{\Phi} + \left(\frac{|m|^2x^2}{1-x^2} - |m| - \frac{|m|^2}{1-x^2}\right)\widetilde{\Phi} = 0.$$

But

$$\frac{|m|^2x^2}{1-x^2} - |m| - \frac{|m|^2}{1-x^2} = \frac{|m|(|m|+1)x^2 - |m|(|m|+1)}{1-x^2} = -|m|(|m|+1),$$

and therefore

$$(1-x^2)\frac{d^2\widetilde{\Phi}}{dx^2} - 2x(|m|+1)\frac{d\widetilde{\Phi}}{dx} + (\lambda - |m|(|m|+1))\widetilde{\Phi} = 0. \tag{4.26}$$

Comparing (4.26) with (4.24), we see that if $P$ solves (4.23), then (4.25) solves (4.21), as claimed.

Therefore, it suffices to solve (4.23). We seek a power series solution of the form

$$P(x) = \sum_{k=0}^{\infty} a_k x^k. \tag{4.27}$$

Plugging (4.27) into (4.23) gives

$$(1-x^2)\sum_{k=0}^{\infty} k(k-1)a_k x^{k-2} - 2x\sum_{k=0}^{\infty} k a_k x^{k-1} + \lambda \sum_{k=0}^{\infty} a_k x^k = 0,$$

or yet, after rearranging some terms,

$$\sum_{k=0}^{\infty} \left((k+2)(k+1)a_{k+2} - (k(k+1)-\lambda)a_k\right) x^k = 0,$$

which implies the following recurrence relation,

$$a_{k+2} = \frac{k(k+1)-\lambda}{(k+1)(k+2)} a_k,\ k = 0, 1, 2, \ldots. \tag{4.28}$$

Relation (4.28) determines all coefficients $a_k$ except for $a_0$ and $a_1$, which remain arbitrary (this is consistent with the fact that we are solving a second order ODE). Furthermore, $a_0$ determines all even coefficients, giving rise to an even power series, while while $a_1$ determines all odd coefficients, giving rise to an odd power series. These two power series, even and odd, are two linearly independent solutions of (4.23).

Next, we investigate the convergence of (4.27). Since it suffices to investigate the convergence of the even and odd expansions separately, as these are two linearly independent solutions, the ratio between two consecutive terms in the expansion is obtained from (4.28), yielding

$$\lim_{k\to\infty} \left| \frac{a_{k+2}x^{k+2}}{a_k x^k} \right| = |x|^2,$$

and thus (4.27) converges for $|x| < 1$ by the ratio test. We need to investigate the case $|x| = 1$ (which corresponds to $\phi = 0$ or $\phi = \pi$). Plugging $x = \pm 1$ into (4.27) gives

$$P(\pm 1) = \pm \sum_{k=0}^{\infty} a_k. \tag{4.29}$$

From (4.28) we have

$$a_{k+2} = \frac{k^2+O(k)}{k^2+O(k)} a_k = \frac{k^2+O(k)}{k^2+O(k)} \frac{(k-2)^2+O(k)}{(k-2)^2+O(k)} a_{k-2} = \cdots = \begin{cases} \frac{k^{k+2}+O(k^{k+1})}{k^{k+2}+O(k^{k+1})} a_0, & k \text{ even}, \\ \frac{k^{k+1}+O(k^k)}{k^{k+1}+O(k^k)} a_1, & k \text{ odd}. \end{cases}$$

It follows that

$$\lim_{k\to\infty} a_k \neq 0,$$

and therefore (4.29) diverges by the divergence test, *unless* (4.27) is in fact a *finite sum*; i.e., *unless* $a_k = 0$ *for all* $k$ *greater than a certain* $\ell$. Hence, we must have, form some non-negative integer $\ell$,

$$a_{\ell+2} = 0 = \frac{\ell(\ell+1)-\lambda}{(\ell+1)(\ell+2)} a_\ell,$$

which implies

$$\lambda = \ell(\ell+1), \tag{4.30}$$

provided that $a_\ell \neq 0$. Relation (4.30) determines $\lambda$, and hence the separation constant $a$ in view of (4.20). The conclusion is that there is a family $\{P_\ell\}$ of solutions to (4.23) parametrized by $\ell = 0, 1, 2, \ldots$. After conveniently choosing $a_0$ and $a_1$ to obtain integer coefficients, the first few $P$'s are

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = 1 - 3x^2,\ P_3(x) = 3x - 5x^3.$$

Since $P_\ell$ is a polynomial of degree $\ell$, from (4.22) it follows that $\Phi = 0$ for $|m| > \ell$. Thus, the values of $m$ are restricted to $|m| \leq \ell$, i.e., the allowed $m$-values depend on $\ell$ and satisfy

$$m \in \big\{ -\ell, -\ell+1, \ldots, -1, 0, 1, \ldots, \ell-1, \ell \big\}. \tag{4.31}$$

We write $m = m_\ell$ when we want to stress this dependence of the allowed values of $m$ on $\ell$. One obtains a family of solutions $\{\Phi_{\ell m_\ell}\}$ to (4.21) parametrized by $\ell$ and $m_\ell$, where $\ell = 0, 1, 2, \ldots$ and $m_\ell$ satisfies (4.31). The first few $\Phi$'s are

$$\Phi_{00}(x) = 1,$$

$$\Phi_{10}(x) = x,\ \Phi_{1,\pm1}(x) = (1-x^2)^{\frac{1}{2}},$$

$$\Phi_{20}(x) = 1 - 3x^2,\ \Phi_{2\pm1}(x) = (1-x^2)^{\frac{1}{2}}x,\ \Phi_{2\pm2}(x) = 1 - x^2,$$

$$\Phi_{30}(x) = 3x - 5x^3,\ \Phi_{3\pm1}(x) = (1-x^2)^{\frac{3}{2}}(1-5x^2),\ \Phi_{3\pm2}(x) = (1-x^2)x,\ \Phi_{3\pm3}(1-x^2)^{\frac{3}{2}}.$$

Finally, it is necessary to rewrite our solutions in terms of the $\phi$ variable. Denoting

$$F_{\ell m_\ell} = \widetilde{\Phi}_{\ell m_\ell},$$

and using $1 - x^2 = \sin^2\phi$,

$$\Phi_{\ell m_\ell}(\phi) = \sin^{|m_\ell|} F_{\ell m_\ell}(\cos\phi),\ \ell = 0, 1, 2, \ldots,\ |m_\ell| \leq \ell. \tag{4.32}$$

Combining (4.13), (4.18), and (4.32) gives

$$Y_{\ell m_\ell}(\phi, \theta) = e^{i m_\ell \theta} \sin^{|m_\ell|}\phi F_{\ell m_\ell}(\cos\phi),\ \ell = 0, 1, 2, \ldots,\ |m_\ell| \leq \ell. \tag{4.33}$$

Notice that, in view of (4.12) and (4.20), $Y_{\ell,m_\ell}$ solves

$$\Delta_{S^2} Y_{\ell m_\ell} = -\ell(\ell+1) Y_{\ell m_\ell},$$

which is an eigenvalue problem for $\Delta_{S^2}$; thus, the spherical harmonics are the eigenfunctions of $\Delta_{S^2}$. We finish this section with some terminology. Equation (4.23) is known as Legendre equation, and its solutions $P_\ell$ are known as Legendre polynomials. The functions $F_{\ell m_\ell}$ are known as associated Legendre functions. The functions $Y_{\ell m_\ell}$ are called spherical harmonics. Legendre functions and spherical harmonics have many important applications in Physics. The interested reader is referred to [1] for details.

4.2.3. *The radial equation.* We now turn our attention to equation (4.11). Using (4.20) and (4.30), equation (4.11) can be written as

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{2\mu}{\hbar^2}\left(E - V(r)\right) R = \ell(\ell+1)\frac{R}{r^2}. \tag{4.34}$$

It is important to stress that the results of section 4.2.2 are general, i.e., they apply to separation of variables to any radially symmetric potential $V = V(r)$. To solve (4.34), on the

other hand, we need to specify the function $V(r)$. We shall assume that $V$ is the potential describing the electromagnetic interaction of an electron with a nucleus. This covers the important case when one is solving the Schrödinger equation describing the evolution of an electron on a hydrogen atom. In this situation, $V$ takes the form

$$V(r) = -\frac{Ze^2}{4\pi\varepsilon_0 r}, \tag{4.35}$$

where $Z$ is the nuclear charge (for example, $Z = 1$ for the hydrogen and $Z = 2$ for an ionized helium atom), $-e$ is the electron charge, where $e = 1.6 \times 10^{-19}\, C$, and $\varepsilon_0$ is the vacuum permitivity whose values is $\varepsilon_0 = 8.85 \times 10^{-12}\, F/m$ (farads per meters). In order to investigate solutions to (4.34) with $V$ given by (4.35), one needs more information about the separation constant $E$.

We claim that $E$ must be real and negative. To see this, multiply equation (4.34) by $r^2R^*$, where $R^*$ is the complex conjugate of $R$, and integrate from 0 to $\infty$:

$$\int_0^\infty R^* \frac{d}{dr}\left(r^2\frac{dR}{dr}\right)\, dr - \frac{2\mu}{\hbar^2}\int_0^\infty V|R|^2 r^2\, dr - \ell(\ell+1)\int_0^\infty |R|^2\, dr = -\frac{2\mu}{\hbar^2}E\int_0^\infty |R|^2 r^2\, dr, \tag{4.36}$$

where we used that $|R|^2 = R^*R$. Integrating by parts the first term,

$$\int_0^\infty R^* \frac{d}{dr}\left(r^2\frac{dR}{dr}\right)\, dr = -\int_0^\infty \frac{dR^*}{dr}\frac{dR}{dr}r^2\, dr + R^* r^2 \frac{dR}{dr}\Big|_0^\infty = -\int_0^\infty \frac{dR^*}{dr}\frac{dR}{dr}r^2\, dr \tag{4.37}$$

where it has been assumed that $R^*$ and $\frac{dR}{dr}$ vanish sufficiently fast at $\infty$. Writing

$$R = R_R + iR_C,$$

where $R_R$ and $R_C$ are real-valued, it comes

$$\frac{dR^*}{dr}\frac{dR}{dr} = (\frac{dR_R}{dr} - i\frac{dR_C}{dr})(\frac{dR_R}{dr} + i\frac{dR_C}{dr}) = \left(\frac{dR_R}{dr}\right)^2 + \left(\frac{dR_C}{dr}\right)^2,$$

and we conclude that $\frac{dR^*}{dr}\frac{dR}{dr}$ is real-valued. But from (4.36) and (4.37) we have

$$E = \frac{\int_0^\infty \frac{dR^*}{dr}\frac{dR}{dr}r^2\, dr + \frac{2\mu}{\hbar^2}\int_0^\infty V|R|^2 r^2\, dr + \ell(\ell+1)\int_0^\infty |R|^2\, dr}{\frac{2\mu}{\hbar^2}\int_0^\infty |R|^2 r^2\, dr}. \tag{4.38}$$

Therefore, since all terms on the right-hand side are real, we conclude that $E$ is real as well. Students should notice that (4.38) gives an explicit expression for $E$ in terms of (the integral of) $R$ and other data of the problem (although we shall derive a much more explicit expression for $E$, see below).

Now that we know that $E$ is real, let us show that it is negative[1]. Let us investigate the behavior of (4.34) for large values of $r$, i.e., $r \gg 1$. Then we can neglect the terms that contain $\frac{1}{r}$ and (4.34) gives, after expanding the terms in $\frac{d}{dr}$,

$$\frac{d^2R}{dr^2} \approx -\frac{2\mu E}{\hbar^2}R. \tag{4.39}$$

[1]It is possible to obtain $E < 0$ by a more delicate analysis of (4.38), but here we employ a simpler argument.

But for $r \gg 1$ we also have the approximation

$$r\frac{d^2R}{dr^2} + \frac{dR}{dr} \approx r\frac{d^2R}{dr^2},$$

so that

$$\frac{d^2(rR)}{dr^2} = r\frac{d^2R}{dr^2} + 2\frac{dR}{dr} \approx r\frac{d^2R}{dr^2}. \tag{4.40}$$

Hence, multiplying (4.39) by $r$ and using (4.40),

$$\frac{d^2(rR)}{dr^2} \approx -\frac{2\mu E}{\hbar^2}(rR).$$

This approximate equation can be easily solved, producing

$$rR \approx e^{\pm\frac{\sqrt{-2\mu E}}{\hbar}r}.$$

If $E \geq 0$, then $R$ is a complex function which satisfies

$$|rR| \approx 1 \text{ for } r \gg 1.$$

Then the integral

$$\int_{\mathbb{R}^3} |\Psi(t,x)|^2 \, dx = \left(\int_0^{2\pi}\int_0^{\pi} |Y(\phi,\theta)|^2 \sin\phi \, d\phi d\theta\right)\left(\int_0^{\infty} |R(r)|^2 r^2 \, dr\right)$$

diverges since $|R(r)|^2 r^2 \approx 1$ for large $r$. Consequently, condition (4.3) fails, and this does not produce a physically sensible solution.

In light of the above arguments, we assume, once and for all, that $E < 0$. In this case, we can define the real constants

$$\beta^2 = -\frac{2\mu E}{\hbar^2}, \tag{4.41}$$

and

$$\gamma = \frac{\mu Z e^2}{4\pi\varepsilon_0\hbar^2\beta}, \tag{4.42}$$

and make the real change of variables

$$\varrho = 2\beta r.$$

With these definitions, equation (4.34), with $V$ given by (4.35), becomes

$$\frac{1}{\varrho^2}\frac{d}{d\varrho}\left(\varrho^2\frac{dR}{d\varrho}\right) + \left(-\frac{1}{4} - \frac{\ell(\ell+1)}{\varrho^2} + \frac{\gamma}{\varrho}\right)R = 0. \tag{4.43}$$

Equation (4.43) will be solved using a power series expansion, but direct application of the method does not work. To see this, try plugging

$$R(\varrho) = \sum_{k=0}^{\infty} a_k \varrho^k$$

into (4.43), obtaining

$$\sum_{k=0}^{\infty} k(k+1)a_k\varrho^{k-2} - \frac{1}{4}\sum_{k=0}^{\infty} a_k\varrho^k - \ell(\ell+1)\sum_{k=0}^{\infty} a_k\varrho^{k-2} + \gamma\sum_{k=0}^{\infty} a_k\varrho^{k-1} = 0.$$

This can be rewritten as

$$-\ell(\ell+1)a_0\varrho^{-2} + ((2-\ell(\ell+1))a_1 + \gamma a_0)\,\varrho^{-1} + \sum_{k=0}^{\infty}\left(((k+3)(k+2)-\ell(\ell+1))\,a_{k+2} + \gamma a_{k+1} - \frac{1}{4}a_k\right)\varrho^k = 0. \tag{4.44}$$

Vanishing of each term order by order implies that $a_0 = 0$, then $a_1 = 0$, and subsequently $a_k = 0$ for any $k$, so $R = 0$. We need, therefore, to try a different approach.

We shall focus on the behavior of (4.43) when $\varrho \gg 1$, in which case the equation simplifies to

$$\frac{1}{\varrho^2}\frac{d}{d\varrho}\left(\varrho^2\frac{dR}{d\varrho}\right) \approx \frac{R}{4}. \tag{4.45}$$

This (approximate) equation can be solved as follows. Look for a solution of the form $e^{A\varrho}$. Plugging into the equation we find $A = -\frac{1}{2}$, i.e., $e^{-\frac{\varrho}{2}}$ is a (approximate) solution of (4.45). This suggests[2] looking for solutions of (4.43) in the form

$$R(\varrho) = e^{-\frac{\varrho}{2}}G(\varrho). \tag{4.46}$$

Plugging (4.46) into (4.43) gives an equation for $G$,

$$\frac{d^2G}{d\varrho^2} + \left(\frac{2}{\varrho} - 1\right)\frac{\partial G}{\partial \varrho} + \left(\frac{\gamma-1}{\varrho} - \frac{\ell(\ell+1)}{\varrho^2}\right)G = 0. \tag{4.47}$$

We seek a solution of the form

$$G(\varrho) = \varrho^s\sum_{k=0}^{\infty}a_k\varrho^k = \sum_{k=0}^{\infty}a_k\varrho^{k+s}, \tag{4.48}$$

where $s$ is to be determined. The term $\varrho^s$ has been included due to the $\frac{1}{\varrho}$ terms in the equation, as these may lead to singular terms that do not fit into a general recurrence relation, as it occurred in (4.44). Notice that the traditional procedure is included in this approach by simply setting $s = 0$.

Plugging (4.48) into (4.47) gives, after some algebra,

$$(s(s+1)-\ell(\ell+1))\,a_0\varrho^{s-2} + \sum_{k=0}^{\infty}\Big(((s+k+1)(s+k+2)-\ell(\ell+1))a_{k+1} - (s+k+1-\gamma)a_k\Big)\varrho^{s+k-1} = 0. \tag{4.49}$$

The vanishing of the term in $\varrho^{s-2}$ requires

$$s(s+1) - \ell(\ell+1) = 0,$$

which has roots $s = \ell$ and $s = -(\ell+1)$. This latter root is rejected on the basis that it does not yield a finite solution when $\varrho \to 0^+$, i.e., $G(\varrho)$ blows up at the origin when $s = -(\ell+1)$ (recall that $\ell$ is non-negative).

[2]The reader may remember that when one solves second order ODEs with constant coefficients, sometimes we have to multiply a solution by a suitable power of the variable in order to produce a particular solution or a second linearly independent solution. What it is being done here resembles that: we have some information about solutions, i.e., that $e^{-\frac{\varrho}{2}}$ solves the equation (in an approximate sense) for large values of $\varrho$. Thus, we try multiplying by $e^{-\frac{\varrho}{2}}$ to construct the full, exact solution.

Using $s = \ell$, one finds from (4.49) the following recurrence relation,

$$a_{k+1} = \frac{k + \ell + 1 - \gamma}{(k + \ell + 1)(k + \ell + 2) - \ell(\ell + 1)} a_k. \tag{4.50}$$

From (4.50) and the ratio test, we see at once that (4.48), with $s = \ell$, converges for all values of $\varrho$.

In order for (4.48) to be an acceptable solution, we also must verify (4.3). From (4.50), it follows that

$$a_{k+1} = \frac{k + \cdots}{k^2 + \cdots} a_k = \frac{1 + \cdots}{k + \cdots} a_k,$$

and

$$a_k = \frac{k - 1 + \cdots}{(k-1)^2 + \cdots} a_{k-1} = \frac{1 + \cdots}{(k-1) + \cdots} a_{k-1},$$

so that

$$\begin{aligned} a_{k+1} &= \frac{1 + \cdots}{k + \cdots} a_k = \frac{1 + \cdots}{k + \cdots} \frac{1 + \cdots}{(k-1) + \cdots} a_{k-1} \\ &= \frac{1 + \cdots}{k(k-1) + \cdots} a_{k-1}. \end{aligned}$$

Continuing this way,

$$a_{k+1} = \frac{1 + \cdots}{k(k-1)(k-2)\cdots(k-j) + \cdots} a_{k-j}.$$

Remembering that

$$e^{\varrho} = \sum_{k=0}^{\infty} \frac{1}{k!} \varrho^k,$$

we see that $G(\varrho)$ is asymptotic to $\varrho^s e^{\varrho}$, i.e., its series expansion behaves very much like the series of $\varrho^{\ell} e^{\varrho}$ (recall that $s = \ell$):

$$G(\varrho) \sim \varrho^{\ell} e^{\varrho},$$

which implies, upon recalling (4.46),

$$R(\varrho) = e^{-\frac{\varrho}{2}} G(\varrho) \sim e^{-\frac{\varrho}{2}} \varrho^{\ell} e^{\varrho} = \varrho^{\ell} e^{\frac{\varrho}{2}},$$

which diverges when $\varrho \to \infty$. As a consequence, (4.3) is not satisfied. This will be the case, *unless* the series (4.48) terminates, i.e., unless $a_k = 0$ for all $k$ greater than a certain $n$. From (4.50), this means

$$k + \ell + 1 - \gamma = 0,$$

i.e.,

$$\gamma = k + \ell + 1.$$

In particular, $\gamma$ has to be an integer,

$$\gamma = n,\ n = \ell + 1, \ell + 2, \ldots.$$

With this, the series terminates at the $(n-(\ell+1))^{\text{th}}$ term, and $G$ is a polynomial of degree $n-1$. Recalling (4.41) and (4.42), we have found the possible values for the separation constant $E = E_n$, namely,

$$E_n = -\frac{\mu Z^2 e^4}{2(4\pi\varepsilon_0)^2\hbar^2 n^2},\ n = 1, 2, 3, \ldots \tag{4.51}$$

We write $R_{n\ell}$ to indicate that $R$ is parametrized by the integers $n$ and $\ell$, with $n = \ell, \ell+1, \ldots$. We can now write, for each $n$, the corresponding $R_{n\ell}$ by using (4.50) to find the polynomial $G = G_{n\ell}$, and then $R_{n\ell}$ via (4.46). Unwrapping all our definitions,

$$R_{n\ell}(r) = e^{-\frac{Zr}{n\alpha_0}} \left(\frac{Zr}{n\alpha_0}\right)^{\ell} G_{n\ell}\left(\frac{Zr}{n\alpha_0}\right),$$

where

$$\alpha_0 = \frac{4\pi\varepsilon_0\hbar^2}{\mu e^2}.$$

In light of (4.10), we see that $\psi$ is also parametrized by $n, \ell$, and $m_\ell$. Instead of thinking of $n$ varying according to $n = \ell, \ell+1, \ldots$, we can equivalently think of $\ell$ as constrained by $\ell = 0, 1, \ldots, n-1$, for each given $n = 1, 2, \ldots$, what is more convenient in order to organize the parameters $n, \ell, m_\ell$. We obtain, therefore, a family of solutions to (4.6),

$$\psi_{n\ell m_\ell} = R_{n\ell} Y_{\ell m_\ell}, \tag{4.52}$$

where

$$\begin{aligned} &n = 1, 2, 3, \ldots, \\ &\ell = 0, 1, 2, \ldots, n-1, \\ &m_\ell = -\ell, -\ell+1, \ldots, 0, \ldots, \ell-1, \ell. \end{aligned} \tag{4.53}$$

Our final solution is then given, in view of (4.4) and (4.7), by

$$\Psi(t, x) = A_{n\ell m_\ell} e^{-\frac{iE_n}{\hbar}t} \psi_{n\ell m_\ell}(x),$$

where $n, \ell$, and $m_\ell$ satisfy (4.53), $E_n$ and $\psi_{n\ell m_\ell}$ are given by (4.51) and (4.52), respectively, and $A_{n\ell m_\ell}$ is a constant (depending on $n$, $\ell$, and $m_\ell$) that ensures (4.2), i.e., $A_{n\ell m_\ell}$ is given by

$$A_{n\ell m_\ell} = \left(\int_{\mathbb{R}^3} |\psi_{n\ell m_\ell}|^2\right)^{-\frac{1}{2}}.$$

The term $e^{-\frac{iE_n}{\hbar}t}$ does not contribute to $|\Psi|^2$ (since $(e^{-\frac{iE_n}{\hbar}t})^*(e^{-\frac{iE_n}{\hbar}t}) = (e^{+\frac{iE_n}{\hbar}t})(e^{-\frac{iE_n}{\hbar}t}) = 1$). It is customary to absorb the constant $A_{n\ell m_\ell}$ into $\psi_{n\ell m_\ell}$, in which case

$$\int_{\mathbb{R}^3} |\psi_{n\ell m_\ell}|^2 = 1.$$

Or course, (4.2) is automatically satisfied in this case.

4.3. **Final comments.** We close with some remarks about the physical meaning of the problem we just described. Readers are referred to [4] for a more thorough physical discussion. Below, we list some the first few $\psi_{n\ell m_\ell}$.

| $n$ | $\ell$ | $m_\ell$ | $\psi_{n\ell m_\ell}$ |
|---|---|---|---|
| 1 | 0 | 0 | $\psi_{100} = \frac{1}{\sqrt{\pi}} \left(\frac{Z}{a_0}\right)^{\frac{3}{2}} e^{-\frac{Zr}{a_0}}$ |
| 2 | 0 | 0 | $\psi_{200} = \frac{1}{4\sqrt{2\pi}} \left(\frac{Z}{a_0}\right)^{\frac{3}{2}} \left(2 - \frac{Zr}{a_0}\right) e^{-\frac{Zr}{2a_0}}$ |
| 2 | 1 | 0 | $\psi_{210} = \frac{1}{4\sqrt{2\pi}} \left(\frac{Z}{a_0}\right)^{\frac{3}{2}} \frac{Zr}{a_0} e^{-\frac{Zr}{2a_0}} \cos\phi$ |
| 2 | 1 | $\pm 1$ | $\psi_{21\pm1} = \frac{1}{8\sqrt{2\pi}} \left(\frac{Z}{a_0}\right)^{\frac{3}{2}} \frac{Zr}{a_0} e^{-\frac{Zr}{2a_0}} \sin\phi e^{\pm i\theta}$ |
| 3 | 0 | 0 | $\psi_{300} = \frac{1}{81\sqrt{3\pi}} \left(\frac{Z}{a_0}\right)^{\frac{3}{2}} \left(27 - 18\frac{Zr}{a_0} + 2\frac{Z^2r^2}{z_0^2}\right) e^{-\frac{Zr}{3a_0}}$ |
| 3 | 1 | 0 | $\psi_{310} = \frac{\sqrt{2}}{81\sqrt{\pi}} \left(\frac{Z}{a_0}\right)^{\frac{3}{2}} \left(6 - \frac{Zr}{a_0}\right) \frac{Zr}{a_0} e^{-\frac{Zr}{3a_0}} \cos\phi$ |
| 3 | 1 | $\pm 1$ | $\psi_{31\pm1} = \frac{\sqrt{2}}{81\sqrt{\pi}} \left(\frac{Z}{a_0}\right)^{\frac{3}{2}} \left(6 - \frac{Zr}{a_0}\right) \frac{Zr}{a_0} e^{-\frac{Zr}{3a_0}} \cos\phi e^{\pm i\theta}$ |
| 3 | 2 | 0 | $\psi_{320} = \frac{1}{81\sqrt{6\pi}} \left(\frac{Z}{a_0}\right)^{\frac{3}{2}} \frac{Z^2r^2}{a_0^2} e^{-\frac{Zr}{3a_0}} (3\cos^2\phi - 1)$ |
| 3 | 2 | $\pm 1$ | $\psi_{32\pm1} = \frac{1}{81\sqrt{\pi}} \left(\frac{Z}{a_0}\right)^{\frac{3}{2}} \frac{Z^2r^2}{a_0^2} e^{-\frac{Zr}{3a_0}} \sin\phi\cos\phi e^{\pm i\theta}$ |
| 3 | 2 | $\pm 2$ | $\psi_{32\pm2} = \frac{1}{262\sqrt{\pi}} \left(\frac{Z}{a_0}\right)^{\frac{3}{2}} \frac{Z^2r^2}{a_0^2} e^{-\frac{Zr}{3a_0}} \sin^2\phi e^{\pm 2i\theta}$ |

It is possible to show that the constants $E_n$, $\ell(\ell+1)$, and $m_\ell$ have important physical interpretation: $E_n$ corresponds to the electron energy, $\ell(\ell+1)$ to the magnitude of its orbital angular momentum, and $m_\ell$ to the projection of the orbital angular momentum onto the $z$-axis. The reader should notice that these quantities cannot be arbitrary, being allowed to take values only on a countable set of multiples of integers. This is a distinctive feature of Quantum Mechanics (we say that the energy and orbital angular momentum are "quantized"). The indices $n$, $\ell$, and $m_\ell$ are called *quantum numbers*.

One-electron atoms with $\ell = 0, 1, 2, 3$ are labeled $s, p, d, f$. In hydrogen and hydrogen-like atoms, this letter is preceded by a number giving the energy level $n$. Thus, the lowest energy state of the hydrogen atom is $1s$; the next to the lowest are $2s$ and $2p$; the next $3s, 3p, 3d$ and so on. These are the so-called "atomic orbitals" that the student is likely to have learned in Chemistry. Remembering that $|\Psi|^2$ is a probability density, what these orbitals represent are "clouds of probability," highlighting the regions of three-dimensional space where it is more likely to find the electron. A few illustrations of the atomic orbitals are given in Figures 1, 2, and 3. These figures were generated with the Mathematica package Visualizing Atomic Orbitals that can be found at

`http://demonstrations.wolfram.com/VisualizingAtomicOrbitals/`

We finish mentioning that in a more detailed treatment of the problem, $\mu$ is not exactly the mass of the particle being described, but rather the reduced mass of the system. This is because, strictly speaking, the electron does not orbit the nucleus, but both orbit the center of mass of the system electron–nucleus. This is very much like the situation of the Earth orbiting the Sun: both bodies move due to their reciprocal gravitational attraction, although the Sun, begin much more massive, barely feels the pull caused by Earth's gravitational field,

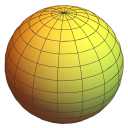


FIGURE 1. An illustration of the orbital $1s$ ($n = 1, \ell = 0, m_\ell = 0$).

FIGURE 2. An illustration of the orbital $2p$ ($n = 2, \ell = 1, m_\ell = 0$).

FIGURE 3. An illustration of the orbital $3d$ ($n = 3, \ell = 2, m_\ell = 0$).

and that is why one usually thinks of the Earth orbiting an standing-still Sun. A similar situation occurs for the nucleus and the electron. We remark, however, that the calculations we presented apply, with no change, to this more accurate situation: we only have to change the value of $\mu$ to be the reduced mass.

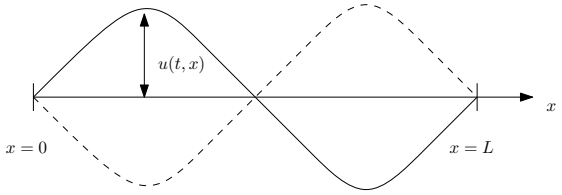
## 5. Separation of variables for the one-dimensional wave equation

Consider the wave equation in one dimension:

$$u_{tt} - c^2 u_{xx} = 0. \quad (c \neq 0)$$

**Notation 5.1.** Whenever a PDE involves the time variable, by the dimension we always mean the spatial dimension. E.g., the one-dimensional wave equation (abbreviated 1d wave equation) is the wave equation for $u = u(t, x)$ with $x \in \mathbb{R}$.

We are interested in the case when the spatial variable belongs to a compact interval, e.g., $0 \leq x \leq L$, for some $L \geq 0$, and $u$ vanishes at the extremities of the interval, i.e., $u(t, 0) = 0 = u(t, L)$. This is the situation describing a string that can vibrate in the vertical direction with its ends fixed, with $u(t, x)$ representing the string amplitude at $x$ at time $t$:




The conditions $u(t, 0) = 0$ and $u(t, L) = 0$ are called boundary conditions because they are conditions imposed on the solution on the boundary of the domain where it is defined. Thus, the problem can be stated as

$$\begin{cases} u_{tt} - c^2 u_{xx} &= 0 \quad \text{in } (0, \infty) \times (0, L) \\ u(t, 0) &= 0 \\ u(t, L) &= 0 \end{cases} \tag{5.1}$$

where the wave equation is defined for all $t \in (0, \infty)$ and $x \in (0, L)$. This is called a boundary value problem (BVP) because it consist of a PDE plus boundary conditions. Sometimes we refer to a boundary value problem simply as PDE. In the HW, you will be asked to show that applying separation of variables we obtain the following family of solutions:

$$u_n(t, x) = \left(a_n \cos\left(\frac{n\pi c}{L} t\right) + b_n \sin\left(\frac{n\pi c}{L} t\right)\right) \sin\left(\frac{n\pi}{L} x\right)$$

where $n = 1, 2, 3, ...$ and $a_n$ and $b_n$ are arbitrary constants. Since the equations is linear, sums fo the above function are solutions, i.e.,

$$\sum_{n=1}^{N} u_n(t, x) = \sum_{n=1}^{N} \left(a_n \cos\left(\frac{n\pi c}{L} t\right) + b_n \sin\left(\frac{n\pi c}{L} t\right)\right) \sin\left(\frac{n\pi}{L} x\right)$$

is also a solution.

Because of this hold for any $N$, we should be able to sum all the way to infinity and still get a solution. In other word, the most general solution to the above boundary value problem is

$$u(t, x) = \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi c}{L} t\right) + b_n \sin\left(\frac{n\pi c}{L} t\right)\right) \sin\left(\frac{n\pi}{L} x\right) \tag{5.2}$$

provided that this expression makes sense, i.e., the series converges.

**Terminology**. It often happens in PDEs that we have situations as the above, i.e., we have a formula for a would-be solution, but we do not know if the formula is in fact well-defined (e.g., we have a series that might not converge, or a function that might not be differentiable, etc.). "Solutions" of this type are called formal solutions. In other words, a formal solution is a candidate for a solution, but extra work must be done or further assumptions made in order to show that they are in fact solutions.

The convergence of the above series cannot be decided without further information about the problem. This is because, as stated, the coefficients $a_n$ and $b_n$ in the formal solution are arbitrary, and it is not difficult to see that we can make different choices of these coefficients in order to make the series converge or diverge.

Therefore, we consider the above BVP (5.1) supplemented by initial conditions, i.e., we assumed given functions $g$ and $h$ defined on $[0, L]$ and look for a solution $u$ such that

$$u(0, x) = g(x), \quad \partial_t u(0, x) = h(x), \quad 0 \leq x \leq L.$$

Similarly to what happens in ODEs, we expect that once initial conditions are given, we will no longer obtain a general solution but rather the unique solutions that satisfies the initial conditions.

**Remark 5.2.** Note that any multiple of the (formal) solution $u$ will also be a (formal) solution. This is encoded in the arbitrariness of $a_n$ and $b_n$, since if we multiple $u$ by a constant $A$, we can simply redefine new coefficients as $\tilde{a}_n = Aa_n, \tilde{b}_n = Ab_n$. This freedom, however, is not present once we consider initial conditions, since if $u(0, x) = g(x), \partial_t u(0, x) = h(x)$ then $Au(0, x) \neq g(x)$, $A\partial_t u(0, x) \neq h(x)$ (unless $A = 1$).

This pervious remark suggests that the coefficients $a_n$ and $b_n$ should be determined from the initial condition. Before investigating this, let us state the full problem. We want to find $u$ such that

$$\begin{cases} u_{tt} - c^2 u_{xx} &= 0 & \text{in } (0, \infty) \times (0, L), c > 0 \\ u(t, 0) = u(t, L) &= 0, & t \geq 0 \\ u(0, x) &= g(x), & 0 \leq x \leq L \\ \partial_t u(0, x) &= h(x), & 0 \leq x \leq L, \end{cases} \tag{5.3}$$

The above problem is called an initial-boundary value problem (IBVP) since it is a PDE with boundary conditions and initial conditions provided, although we sometimes call it simply a PDE.

Two initial conditions are prescribed, i.e., $u(0, x)$ and $\partial_t u(0, x)$, because the wave equation is second order in time. Note that $g$ and $h$ have to satisfy the following compatibility conditions:

$$g(0) = g(L) = h(0) = h(L) = 0. \tag{5.4}$$

We have already derived a formal solution to the wave equation satisfying the boundary conditions (5.2). It remains to investigate the initial conditions. Plugging $t = 0$:

$$u(0, x) = g(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L} x\right).$$

Differentiating $u$ with respect to $t$ and plugging $t = 0$:

$$\partial_t u(0, x) = h(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \sin\left(\frac{n\pi}{L} x\right).$$

Since $g$ and $h$ are in principle arbitrary, the above is essential asking whether it is possible to write an arbitrary function on $[0, L]$ as a series of sine functions with suitable coefficients. Or, rephrasing the equation in a more appropriate form, we are asking: *what are the functions on* $[0, L]$ *that can be written as a convergent series of sine functions with suitable coefficients*? The functions for which this is true will provide us with a class of functions for which the above initial-boundary problem (IBVP) admits a solution.

The subject that investigates questions of this type is known as Fourier series. We will now make a digression to study Fourier series. After that, we will return to the wave equation.

## 6. Fourier Series

We begin with the definition of Fourier series.

**Definition 6.1.** Let $I = (-L, L)$ or $[-L, L]$, $L > 0$, and $f : I \to \mathbb{R}$ be integrable on $I$. The Fourier series of $f$, denoted $F.S.\{f\}$, is the series

$$F.S.\{f\}(x) := \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right), \tag{6.1}$$

where the coefficients $a_n$ and $b_n$ are given by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \quad n = 0, 1, 2, ... \tag{6.2}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \quad n = 1, 2, 3, ... \tag{6.3}$$

The coefficients $a_n$ and $b_n$ are called Fourier coefficients.

**Remark 6.2.**

(1) $F.S.\{f\}$ is a series constructed out of $f$. We are not claiming that $F.S.\{f\} = f$. In fact, at this point we are not evening claiming that $F.S.\{f\}$ converges (although we want to find conditions for which it converges, and for which $F.S.\{f\} = f$ ).
(2) The Fourier coefficients are well defined in view of the integrability of $f$.
(3) We introduced Fourier series for functions defined on an interval $[-L, L]$. This set-up is slightly different than what we encountered above for the wave equation, where we worked on the interval $[0, L]$. We will relate Fourier series on $[-L, L]$ with functions defined on $[0, L]$ later on.
(4) The Fourier series is a series of sine and cosine. The situation discussed above in the wave equation is a particular case where only sine is present (i.e., $a_n = 0$).

**Example 6.3.** Find the Fourier series of

$$f(x) = \begin{cases} -1, & -\pi \le x < 0 \\ 1, & 0 \le x \le \pi. \end{cases}$$

We compute:

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\mathrm{d}x = 0 \quad \text{(even-odd functions)} \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} \underbrace{f(x)}_{=1}\sin(nx)\mathrm{d}x \\ &= \frac{2}{\pi}\left(-\frac{\cos(nx)}{n}\right)\Big|_0^{\pi} = \frac{2}{\pi}\left(\frac{1}{n} - \frac{(-1)^n}{n}\right) \\ &= \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd}. \end{cases} \end{aligned}$$

Thus:

$$F.S.\{f\}(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n}\right)\sin(nx)$$

$$= \frac{4}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + ...\right).$$

Note that $f(0) = 1$ but $F.S.\{f\}(0) = 0$, so $F.S.\{f\} \neq f$.

**Example 6.4.** Find the Fourier series of $f(x) = |x|$, $-1 \leq x \leq 1$. Compute:

$$a_0 = \int_{-1}^{1} f(x)\mathrm{d}x = 2\int_0^1 x\mathrm{d}x = 1,$$
$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\mathrm{d}x = 2\int_0^1 x\cos(n\pi x)\mathrm{d}x = \frac{2}{\pi^2 n^2}((-1)^n - 1), \quad n \in \mathbb{N}$$
$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x)\mathrm{d}x = 0 \quad \text{(even-odd)}.$$

Thus

$$F.S.\{f\}(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}((-1)^n - 1)\cos(n\pi x)$$
$$= \frac{1}{2} - \frac{4}{\pi^2}\left(\cos(\pi x) + \frac{1}{9}\cos(3\pi x) + \frac{1}{25}\cos(5\pi x) + ...\right).$$

6.1. **Piecewise Functions.** We begin with some definitions.

**Definition 6.5.** Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is called $k$-times continuously differentiable if all its derivatives up to order $k$ exists and are continuous. We denote by $C^k(I)$ the space of all $k$-times continuously differentiable functions on $I$. Note that $C^0(I)$ is the space of continuous functions on $I$. We denote by $C^\infty(I)$ the space of infinitely many times differentiable functions on $I$. Sometimes we say simply that "$f$ is $C^k$" to mean that $f \in C^k(I)$. We write simply $C^k$ for $C^k(I)$ if $I$ is implicitly understood. $C^\infty$ functions are also called smooth functions.

**Example 6.6.** $e^x \in C^\infty(\mathbb{R})$, $|x| \in C^0(\mathbb{R})$. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is $C^0$, it is differentiable, but it is *not* $C^1$. This is because $f'(x)$ exists for every $x$ (including $x = 0$) but $f'$ is not continuous at $x = 0$.

**Remark 6.7.** Note that $C^k(I) \subset C^\ell(I)$ if $k > \ell$ and $C^\infty(I) = \bigcap_{k=0}^{\infty} C^k(I)$.

**Definition 6.8.** Let $I \subset \mathbb{R}$ be an interval. We say that $f : I \to \mathbb{R}$ is a piecewise $C^k$ function if $f$ is $C^k$ except possibly at a countable number of isolated points.

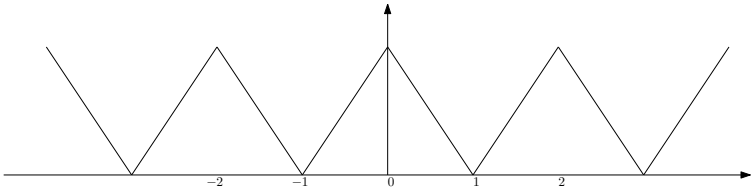
**Example 6.9.**

(1) The functions $|x|$ and

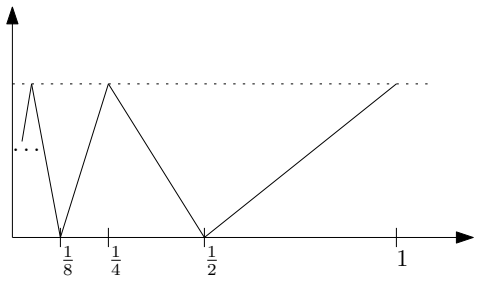
$$f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

are piecewise smooth ($C^\infty$) functions.

(2) Below is piecewise $C^\infty$ function.




(3) The function $f : [0, 1] \to \mathbb{R}$ given by




is not piecewise $C^k$ because the set of points where it fails to be $C^k$ are not isolated.

6.2. **Convergence of Fourier Series.**

**Notation 6.10.** We denote by $f(x^+)$ and $f(x^-)$ the right and left values of $f$ at $x$, defined by

$$f(x^+) = \lim_{h\to 0^+} f(x+h), \quad f(x^-) = \lim_{h\to 0^-} f(x+h).$$

If $f$ is continuous at $x$, then $f(x^+) = f(x^-) = f(x)$, but otherwise these values might differ.

**Example 6.11.** Consider the unit step function,

$$H(t) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0. \end{cases}$$

Then $H(0^+) = 1$ and $H(0^-) = -1$.

**Theorem 6.12.** *Let $f$ be a piecewise $C^1$ function on $[-L, L]$. Then, for any $x \in (-L, L)$:*

$$F.S.\{f\}(x) = \frac{1}{2}(f(x^+) + f(x^-)), \tag{6.4}$$

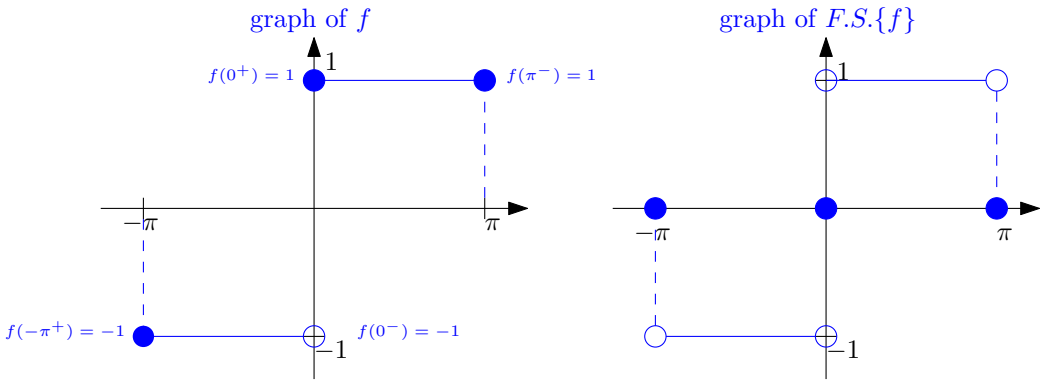
*and*

$$F.S.\{f\}(\pm L) = \frac{1}{2}(f(-L^+) + f(L^-)). \tag{6.5}$$

*In particular, $F.S.\{f\}$ converges.*

From the above theorem, we see that $F.S.\{f\}(x) = f(x)$ when $f$ is continuous at $x$. Thus, if $f$ is piecewise $C^1$ and $C^0$, we have:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right). \tag{6.6}$$




**Example 6.13.** We graph

$$f(x) = \begin{cases} -1, & -\pi \le x < 0 \\ 1, & 0 \le x \le \pi \end{cases}$$

and $F.S.\{f\}(x)$ below (note that $f$ is piecewise $C^1$)

**Example 6.14.** Since $|x|$ is continuous and piecewise $C^1$:

$$|x| = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left((-1)^n - 1\right)\cos(n\pi x).$$

Next, we consider the differentiation and integration of Fourier series term by term.

**Theorem 6.15.** *Let $f$ be a piecewise $C^2$ and continuous function on $[-L, L]$, and assume that $f(-L) = f(L)$. Then, the Fourier series of $f'$ can be obtained from that of $f$ by differentiation term-by-term. More precisely, writing*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right),$$

*we have*

$$\begin{aligned} F.S.\{f'\}(x) &= \sum_{n=1}^{\infty}\left(a_n\left(\cos\left(\frac{n\pi x}{L}\right)\right)' + b_n\left(\sin\left(\frac{n\pi x}{L}\right)\right)'\right) \\ &= \sum_{n=1}^{\infty}\left(\frac{-a_n n\pi}{L}\sin\left(\frac{n\pi x}{L}\right) + \frac{b_n n\pi x}{L}\cos\left(\frac{n\pi x}{L}\right)\right). \end{aligned}$$

*In particular, if $f'$ is continuous at $x$, we have*

$$f'(x) = \sum_{n=1}^{\infty}\frac{n\pi}{L}\left(-a_n \sin\left(\frac{n\pi x}{L}\right) + b_n \cos\left(\frac{n\pi x}{L}\right)\right).$$

**Example 6.16.** To see that we cannot always differentiate a Fourier series term by term, consider $f(x) = x$, $-\pi \le x \le \pi$. Its Fourier series is

$$F.S.\{f\}(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx),$$

which converges for any $x$, but the term-by-term differentiated series, which is

$$2\sum_{n=1}^{\infty}(-1)^{n+1}\cos(nx)$$

diverges for every $x$.

**Theorem 6.17.** *Let $f$ be piecewise continuous on $[-L, L]$ with Fourier series*

$$F.S.\{f\}(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right). \tag{6.7}$$

*Then, for any $x \in [-L, L]$:*

$$\int_{-L}^{x} f(t)\mathrm{d}t = \int_{-L}^{x}\frac{1}{2}a_0\mathrm{d}t + \sum_{n=1}^{\infty}\int_{-L}^{x}\left(a_n\cos\left(\frac{n\pi t}{L}\right) + b_n\sin\left(\frac{n\pi t}{L}\right)\right)\mathrm{d}t. \tag{6.8}$$

6.3. **Some Intuition Behind Fourier Series.** Let us make some comments about the way the Fourier series is defined. Given $f$ defined on $[-L, L]$, our goal is to write:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right)$$

Let us make an analogy with the following problem: given a vector $v \in \mathbb{R}^n$, we want to write

$$v = \sum_{i=1}^{n} c_i e_i,$$

where $\{e_i\}_{i=1}^n$ is an orthogonal basis of $\mathbb{R}^n$. In other words, we have to find the coefficients $c_i$. Since the vectors $e_i$ are orthogonal

$$e_i \cdot e_j = 0 \text{ if } i \neq j,$$

where $\cdot$ is the dot product, a.k.a. inner product of vectors. Thus, for each $j = 1, ..., n$:

$$e_j \cdot v = \sum_{i=1}^{n} c_i e_j \cdot e_i = c_j e_j \cdot e_j \Longrightarrow c_j = \frac{v \cdot e_j}{e_j \cdot e_j}.$$

We want to do something similar to find the Fourier coefficients $a_n$ and $b_n$. Consider the functions

$$E_0(x) = \frac{1}{2}, \quad E_n(x) = \cos\left(\frac{n\pi x}{L}\right), \quad \tilde{E}_n(x) = \sin\left(\frac{n\pi x}{L}\right) \quad n = 1, 2, ....$$

Then:

$$f = a_0 E_0 + \sum_{n=1}^{\infty}(a_n E_n + b_n \tilde{E}_n).$$

This is very similar to the case in $\mathbb{R}^n$. In fact, the space of piecewise $C^k$ is a vector space, so (6.3) is an equality between vectors, although $C^k$ is an infinite dimensional vector space so we need a basis with infinitely many vectors.

To find the Fourier coefficients the series same way we found the coefficients $c_j$ above, we need the analogue of the dot product for functions. It cannot be the usual product of functions, since the product of two functions is another function, whereas the dot product

of two vectors is not another vector but a number. We also want our "dot product" for functions to have all the standard properties of the dot product of vectors. The relevant product for functions is defined below.

**Definition 6.18.** Let $I \subset \mathbb{R}$ be an interval. The $L^2$ inner product, or simply inner product, of two functions $f, g : I \to \mathbb{R}$ is defined as

$$\langle f, g\rangle_{L^2} := \int_I f(x)g(x)\mathrm{d}x \tag{6.9}$$

whenever the integral on the RHS is well-defined. We often write $\langle \cdot, \cdot \rangle$ for $\langle \cdot, \cdot \rangle_{L^2}$. The $L^2$ norm, or simply norm, of $f : I \to \mathbb{R}$ is defined as

$$\|f\|_{L^2} := \sqrt{\langle f, f\rangle}. \tag{6.10}$$

We sometimes write $\| \cdot \|$ for $\| \cdot \|_{L^2}$. We also write $\langle \cdot, \cdot \rangle_{L^2(I)}$ and $\| \cdot \|_{L^2(I)}$ if we want to emphasize the interval $I$.

It is a simple exercise to show that $\langle \cdot, \cdot \rangle_{L^2}$ has all the following properties, which are similar to the properties of the dot product:

(1) $\langle f, g\rangle \in \mathbb{R}$ when defined
(2) $\langle f, g\rangle = \langle g, f\rangle$
(3) $\langle f, ag + bh\rangle = a\langle f, g\rangle + b\langle f, h\rangle$, $a, b \in \mathbb{R}$, $f, g, h$ functions
(4) $\langle f, 0\rangle = 0$
(5) $\langle f, f\rangle \geq 0$. In particular, $\| \cdot \|_{L^2}$ is a real number if $\langle f, f\rangle < \infty$.

**Remark 6.19.** The dot product has the property $v \cdot v = 0 \Rightarrow v = 0$. This is not true for $\langle \cdot, \cdot \rangle_{L^2}$, as the example

$$f(x) = \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$

shows. However, if $f$ is continuous, then it is true that $\langle f, f\rangle_{L^2} = 0 \Rightarrow f = 0$.

Consider now $I = [-L, L]$ and let us go back to (6.3). A simple computation shows that $\langle E_n, E_m\rangle = 0$ if $n \neq m$, $\langle \tilde{E}_n, \tilde{E}_m\rangle = 0$ if $n \neq m$, $\langle E_n, \tilde{E}_m\rangle = 0$, $\langle \tilde{E}_n, \tilde{E}_n\rangle = L$,

$$\langle E_n, E_n\rangle = \begin{cases} \frac{L}{2}, & n = 0 \\ L, & n > 0. \end{cases}$$

Taking the inner product of (6.3) with $E_m, \tilde{E}_m, m \geq 1$, and $E_0$, gives:

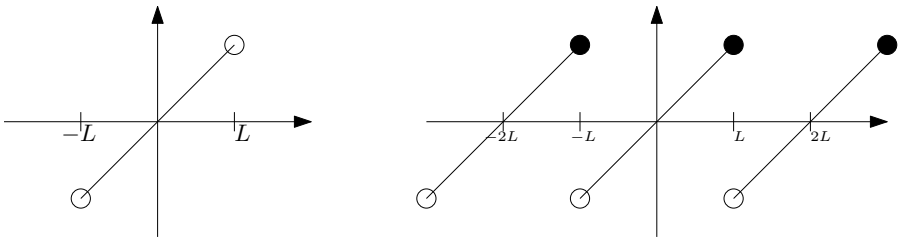
$$\begin{aligned}
\langle f, E_m\rangle &= a_0\langle E_0, E_m\rangle + \sum_{n=1}^{\infty}(a_n\langle E_n, E_m\rangle + b_n\langle \tilde{E}_n, E_m\rangle) \\
&= a_m\langle E_m, E_m\rangle = a_m L \Rightarrow a_m = \frac{\langle f, E_m\rangle}{L} \\
\langle f, E_0\rangle &= a_0\langle E_0, E_0\rangle + \sum_{n=1}^{\infty}(a_n\langle E_n, E_0\rangle + b_n\langle \tilde{E}_n, E_0\rangle) \\
&= a_0\langle E_0, E_0\rangle = a_0\frac{L}{2} \Rightarrow a_0 = \frac{2}{L}\langle f, E_0\rangle \\
\langle f, \tilde{E}_m\rangle &= a_0\langle E_0, \tilde{E}_m\rangle + \sum_{n=1}^{\infty}(a_n\langle E_n, \tilde{E}_m\rangle + b_n\langle \tilde{E}_n, \tilde{E}_m\rangle)
\end{aligned}$$

$$= b_m \langle \tilde{E}_m, \tilde{E}_m \rangle = b_m L \Rightarrow b_m = \frac{\langle f, \tilde{E}_m \rangle}{L}$$

Writing explicitly $\langle \cdot, \cdot \rangle$ in terms of an integral and using the definitions of $E_n, \tilde{E}_n$, we see that the expressions we found for $a_n, b_n$ are exactly the Fourier coefficients.

6.4. **The Fourier series of series of periodic functions, and the Fourier series of functions on** $[0, L]$**.** Suppose that $f$ is defined on $\mathbb{R}$ and has period $2L$, i.e., $f(x) = f(x+2L)$ for all $x$. Thus, all information about $f$ is determined by its values on $[-L, L]$. We can defined the Fourier series for $f$ as a function on $[-L, L]$, and all the previous results are immediately adapted to this case.

Moreover, given a function $(-L, L)$, we can extend it to a periodic function on $\mathbb{R}$ and consider its Fourier series (note, however, that this extension is not unique). This is illustrated in the picture below:




Consider now a function $f$ defined on $[0, L]$. We define its cosine Fourier series by

$$F.S.^{\cos}\{f\}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi L}{x}\right), \quad x \in [0, L], \tag{6.11}$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x.$$

Extend $f$ to an even function on $[-L, L]$ by

$$\tilde{f}(x) = \begin{cases} f(x), & 0 \le x \le L, \\ f(-x), & -L \le x < 0. \end{cases} \tag{6.12}$$

The Fourier coefficients of $\tilde{f}$ are

$$\tilde{a}_n = \frac{1}{L} \int_{-L}^{L} \tilde{f}(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x = a_n, \tag{6.13}$$

$$\tilde{b}_n = \frac{1}{L} \int_{-L}^{L} \tilde{f}(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = 0, \tag{6.14}$$

where we use that $\tilde{f}$ is even. Thus, for $x \in [0, L]$

$$F.S.\{\tilde{f}\}(x) = F.S.^{\cos}\{f\}(x). \tag{6.15}$$

In other words, the cosine Fourier series of $f : [0, L] \to \mathbb{R}$ equals the restriction to $[0, L]$ of the Fourier series of the even extension of $f$.

Similarly, we define the sine Fourier series of $f : [0, L] \to \mathbb{R}$ by

$$F.S.^{\sin}\{f\}(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \tag{6.16}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x.$$

Letting $\tilde{f}$ be an odd extension of $f$,

$$\tilde{f}(x) = \begin{cases} f(x), & 0 \le x \le L, \\ -f(-x), & -L \le x < 0. \end{cases} \tag{6.17}$$

We find the Fourier coefficients of $\tilde{f}$ to be

$$\tilde{a}_n = \frac{1}{L}\int_{-L}^{L} \tilde{f}(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = 0, \tag{6.18}$$

$$\tilde{b}_n = \frac{1}{L}\int_{-L}^{L} \tilde{f}(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x = b_n, \tag{6.19}$$

thus $F.S.\{\tilde{f}\}(x) = F.S.^{\sin}\{f\}(x), x \in [0, L]$. In other words, the sine Fourier series of $f : [0, L] \to \mathbb{R}$ equals the restriction to $[0, L]$ of the Fourier series of the odd extension of $f$.

We conclude that the theorems on convergence, differentiation, and integration of Fourier series are immediately applicable to the sine and cosine Fourier series.

6.5. **Back to the wave equation.** We are now ready to discuss the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} &= 0 & \text{in } (0,\infty)\times(0,L), c > 0 \\ u(t,0) = u(t,L) &= 0, & t \ge 0 \\ u(0,x) &= g(x), & 0 \le x \le L \\ \partial_t u(0,x) &= h(x), & 0 \le x \le L, \end{cases} \tag{6.20}$$

where $g$ and $h$ are given functions satisfying the compatibility conditions

$$g(0) = g(L) = 0 = h(0) = h(L).$$

We saw that a formed solution to this problem is given by:

$$u(t,x) = \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi ct}{L}\right) + b_n\sin\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right) \tag{6.21}$$

where $a_n$ and $b_n$ are to be determined by

$$g(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right),$$

and

$$h(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \sin\left(\frac{n\pi x}{L}\right).$$

The last two expressions mean that $g$ and $h$ equal their sine Fourier series, with Fourier coefficients given by $a_n$ and $\frac{n\pi c}{L}b_n$, respectively. These equalities will in fact be true if we make suitable assumptions on $g$ and $h$. Let us assume that $g$ and $h$ are $C^2$ functions. Then,

from the previous theorems for Fourier series, we know that $g$ and $h$ equal their sine Fourier series, and the coefficients $a_n$ and $b_n$ are given by

$$a_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \quad b_n = \frac{2}{n\pi c}\int_0^L h(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x. \tag{6.22}$$

Our assumptions on $g$ and $h$ allow us to compute the coefficients $a_n$ and $b_n$. We will have to develop a few more tools before we are able to show that (6.21) is in fact a solutions. However, we summarize the result here; its proof will be postponed (in fact, it will be assigned as the HW after more background is developed).

**Theorem 6.20.** *Consider the problem (5.3) and assume that $g$ and $h$ are $C^2$ functions such that*

$$g(0) = g(L) = 0 = h(0) = h(L),$$
$$g''(0) = g''(L) = 0 = h''(0) = h''(L).$$

*Then a solution to (5.3) is given by (6.21), where $a_n$ and $b_n$ are given by (6.22).*

**Remark 6.21.** We will explain the assumptions involving second derivatives of $g$ and $h$ when we prove this theorem.

## 7. The 1d wave equation in $\mathbb{R}$

We now consider the problem for $u = u(t, x)$:

$$\begin{cases} u_{tt} - c^2 u_{xx} &= 0 \qquad \text{in } (0,\infty) \times (\infty, \infty), c > 0 \\ u(0,x) &= u_0(x), \quad -\infty \le x \le \infty \\ \partial_t u(0,x) &= u_1(x), \quad -\infty \le x \le \infty, \end{cases} \tag{7.1}$$

This is an initial-value problem for the wave equation (IVP). Compared to the initial-boundary value problem we studied earlier, we see that now $x \in \mathbb{R}$, so there are no boundary conditions. This initial-value problem is also known as the Cauchy problem for the wave equation, a terminology that we will explain in more detail later on. We refer to the functions $u_0$ and $u_1$ as (initial) data for the Cauchy problem. A solution to this Cauchy problem is a function that satisfies the wave equation and the initial conditions.

We had defined the spaces $C^k(I)$ for an interval $I \subset \mathbb{R}$. For functions of two variables, we can similarly define $C^k(\mathbb{R}^2)$, which we will use here. We will define general $C^k$ spaces for functions several variables later on.

**Proposition 7.1.** *Let $u \in C^2(\mathbb{R}^2)$ be a solution to the 1d wave equation. Then, there exists functions $F, G \in C^2(\mathbb{R})$ such that*

$$u(t,x) = F(x+ct) + G(x-ct).$$

*Proof.* Set $\alpha := x + ct, \beta := x - ct$, so that $t = \frac{1}{2c}(\alpha - \beta), x = \frac{1}{2}(\alpha + \beta)$, and

$$v(\alpha, \beta) := u(t(\alpha,\beta), x(\alpha,\beta)).$$

Then, from $u(t,x) = v(\alpha(t,x), \beta(t,x))$ we find

$$\begin{aligned} u_t &= v_\alpha \alpha_t + v_\beta \beta_t = c v_\alpha - c v_\beta, \\ u_{tt} &= c v_{\alpha\alpha}\alpha_t + c v_{\alpha\beta}\beta_t - c v_{\beta\alpha}\alpha_t - c v_{\beta\beta}\beta_t \\ &= c^2 v_{\alpha\alpha} - c^2 v_{\alpha\beta} - c^2 v_{\beta\alpha} + c^2 v_{\beta\beta}, \\ u_x &= v_\alpha \alpha_x + v_\beta \beta_x = v_\alpha + v_\beta, \\ u_{xx} &= v_{\alpha\alpha}\alpha_x + v_{\alpha\beta}\beta_x + v_{\beta\alpha}\alpha_x + v_{\beta\beta}\beta_x \\ &= v_{\alpha\alpha} + v_{\alpha\beta} + v_{\beta\alpha} + v_{\beta\beta}. \end{aligned}$$

Thus, $0 = u_{tt} - c^2 u_{xx} = -4c^2 v_{\alpha\beta}$, where used that $v_{\alpha\beta} = v_{\beta\alpha}$ since $v$ is $C^2$ (because $u$ is $C^2$ and the change of coordinates $(t,x) \mapsto (\alpha,\beta)$ is $C^\infty$). Thus, in $(\alpha,\beta)$ coordinates the wave equation reads: $v_{\alpha\beta} = 0$.

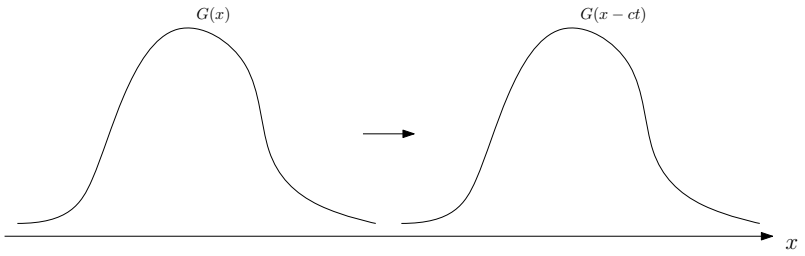
Therefore, $(v_\alpha)_\beta = 0$ implies that $v_\alpha$ is a function of $\alpha$ only: $v_\alpha(\alpha,\beta) = f(\alpha)$ for some $C^1$ function $f$. Integrating w.r.t. $\alpha$ gives

$$v(\alpha,\beta) = \int f(\alpha) \mathrm{d}\alpha + G(\beta),$$

for some function $G$. Note that $F := \int f(\alpha)\mathrm{d}\alpha$ is $C^2$, thus so is $G$. Therefore, $v(\alpha,\beta) = F(\alpha) + G(\beta)$, and in $(t,x)$ coordinates: $u(t,x) = F(x+ct) + G(x-ct)$. □

The above formula has a clear physical interpretation. At $t = 0$, $u(0,x) = F(x) + G(x)$. For each $t > 0$, the graph of $G(x - ct)$ is the graph of $G(x)$ moved of units to the right, so the graph of $G(x)$ is moving to the right with speed $c$. $G(x-ct)$ is called a forward wave. Similarly, the graph of $F(x)$ is moving to the left and $F(x+ct)$ is called a backward wave.

The general solution is thus a sum (or a superposition) of a forward and backward wave, and we see that the constant $c$ is indeed the speed of propagation of the wave.




**Notation 7.2.** Having found the interpretation of the constant $c$, we will often set $c = 1$.

**Proposition 7.3.** *Let $u \in C^2([0,\infty) \times \mathbb{R})$ be a solution to the Cauchy problem for the 1d wave equation with data $u_0, u_1$. Then*

$$u(t,x) = \frac{u_0(t+x) + u_0(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)\mathrm{d}y. \tag{7.2}$$

*This formula is known as D'Alembert's formula.*

*Proof.* Note that $u_0 \in C^2, u_1 \in C^1$. From

$$u(t,x) = F(x+t) + G(x-t)$$

in the pervious result, we have

$$u(0,x) = F(x) + G(x) = u_0(x),$$
$$u_t(0,x) = F'(x) - G'(x) = u_1(x).$$

Integrating this last equality:

$$F(x) - G(x) = \int_0^x u_1(y)\mathrm{d}y + \underbrace{C}_{F(0)-G(0)},$$

adding to $u(0,x)$ :

$$F(x) = \frac{1}{2}u_0(x) + \frac{1}{2}\int_0^x u_1(y)\mathrm{d}y + \frac{C}{2}.$$

Plugging back into $u(0,x)$ :

$$G(x) = \frac{1}{2}u_0(x) - \frac{1}{2}\int_0^x u_1(y)\mathrm{d}y - \frac{C}{2}.$$

Replacing $x \mapsto x+t$ in $F$ and $x \mapsto x-t$ in $G$ and adding gives the formula. □

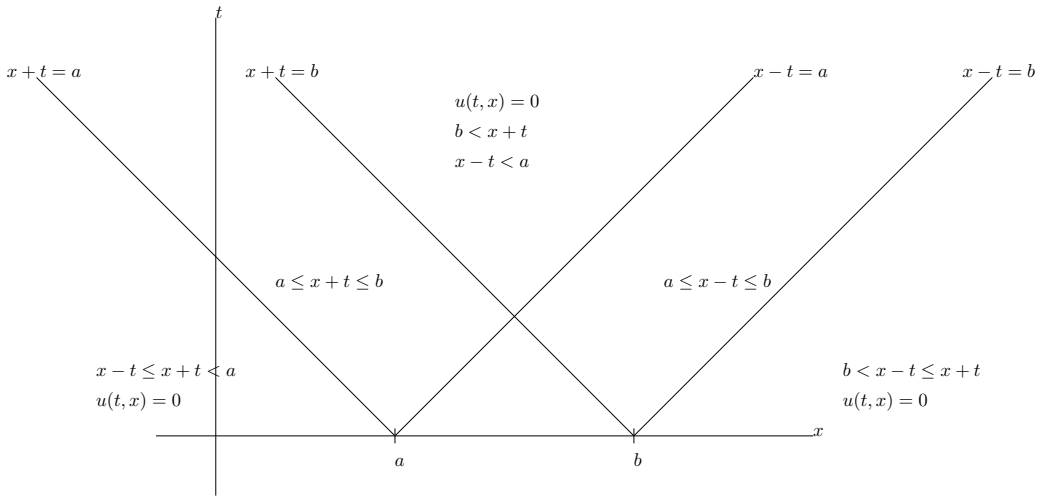
The last two propositions derived formulas for $C^2$ solutions of the wave equation given such a solution. The next result shows that solutions actually exist.

**Theorem 7.4** (∃! of 1d wave)**.** *Let $u_0 \in C^2(\mathbb{R})$ and $u_1 \in C^1(\mathbb{R})$. Then there exists a unique $u \in C^2([0,\infty)\times\mathbb{R})$ that solves the Cauchy problem for the 1d wave equation with data $u_0, u_1$. Moreover, $u$ is given by D'Alembert's formula.*

*Proof.* Given two $C^2([0,\infty) \times \mathbb{R})$ solutions, both satisfy D'Alembert's formula (with the same $u_0, u_1$) thus they are equal, establishing uniqueness. To prove existence, defined $u$ by D'Alembert's formula. Then $u \in C^2([0,\infty) \times \mathbb{R})$ since $u_0 \in C^2$ and $u_1 \in C^1$, and by construction $u$ satisfies the wave equation and the initial conditions. □

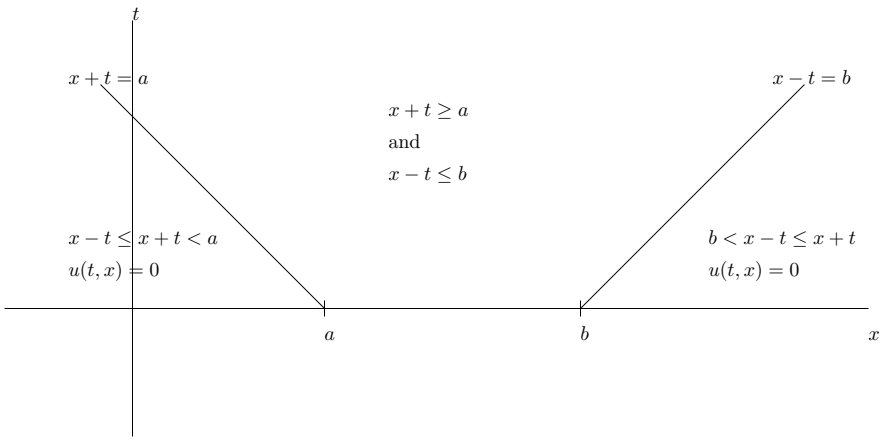
**Definition 7.5.** The lines $x + t =$ constant and $x - t =$ constant in the $(t, x)$ plane (or $x + ct =$ constant, $x - ct =$ constant for $c \neq 1$) are called the characteristics (or characteristic curves) of the wave equation. They (and their generalizations to higher dimensions) are very important to understand solutions to the wave equation, as we will see.

7.1. **Regions of influence for the 1d wave equation.** Suppose $u_1 = 0$ and $u_0(x) = 0$ for $x \notin [a, b]$. Since $u_0(x + t)$ and $u_0(x - t)$ are constant along the lines $x + t =$ constant and $x - t =$ constant, respectively, we see that $u(t, x) \neq 0$ only possibly for points $(t, x)$ that lie int he region determined by the region lying between the characteristics emanating from $a$ and $b$ as indicated in the figure:



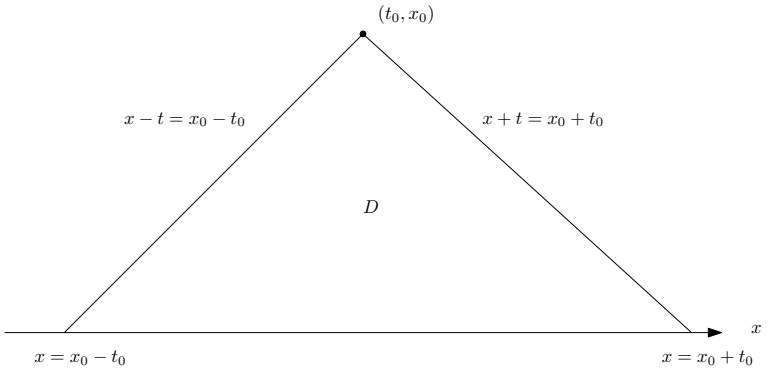

**Notation 7.6.** Although we ordered the coordinates as $(t, x)$, we will often draw the $(t, x)$ plane with the $x$-axis on the horizontal.

Suppose now that $u_0 = 0$ and that $u_1(x) = 0$ for $x \notin [a, b]$. Then $\int_{x-t}^{x+t} u_1(y)\mathrm{d}y = 0$ whenever we have $[x - t, x + t] \cap [a, b] =$, i.e., if $x + t < a$ or $x - t > b$. Therefore, $u(t, x) \neq 0$ possibly only in the region $\{x + t \geq a\} \cap \{x - t \leq b\}$, as depicted in the figure




For general $u_0$ and $u_1$, we can therefore precisely track how the values of $u(t, x)$ are influenced by the values of the initial conditions. It follows that the values of the data on an interval $[a, b]$ can only affect the values of $u(t, x)$ for $(t, x) \in \{x + t \geq a\} \cap \{x - t \leq b\}$. This

reflects the fact that waves travel at a finite speed. The regions $(t,x) \in \{x+t \geq a\} \cap \{x-t \leq b\}$ is called the domain of influence of $[a,b]$. Consider now a point $(t_0,x_0)$ and $u(t_0,x_0)$. Let $D$ be the triangle with vertex $(t_0,x_0)$ determined by $x+t = x_0+t_0, x-t = x_0-t_0$, and $t=0$:




Then

$$u(t_0,x_0) = \frac{u_0(x_0+t_0)+u_0(x_0-t_0)}{2} + \frac{1}{2}\int_{x_0-t_0}^{x_0+t_0} u_1(y)\mathrm{d}y.$$

and we see that $u(t_0,x_0)$ is completely determined by the values of the initial data on the interval $[x_0-t_0, x_0+t_0]$. The region $D$ is called the (past) domain of dependence of $(t_0,x_0)$.

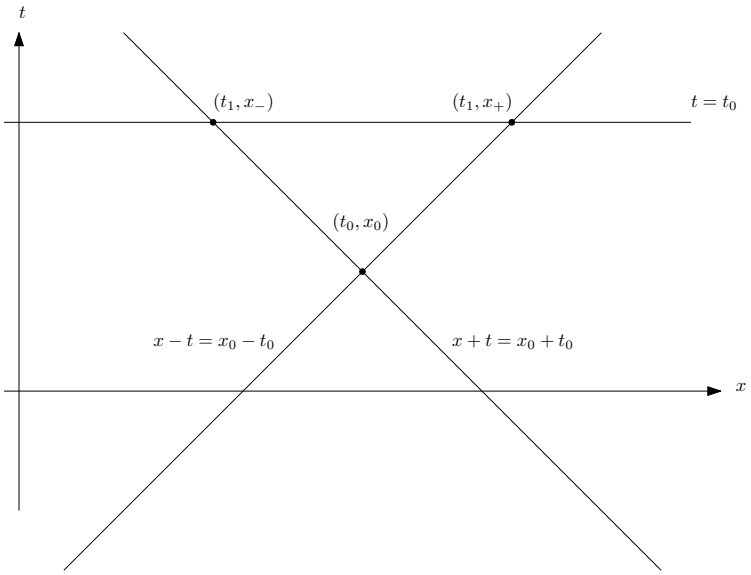
7.2. **Generalized solutions.** Note that the RHS of D'Alembert's formula (7.2) makes sense when $u_0$ and $u_1$ are piecewise functions. This motivates the following definition.

**Definition 7.7.** Let $u_0$ be a piecewise $C^2$ function and $u_1$ a piecewise $C^1$ function. Then $u$ given by D'Alembert's formula is called a generalized solution to the wave equation. If $u_0$ and $u_1$ are $C^2$ and $C^1$ functions, respectively, then $u$ is called a classical solution. When $u$ is a generalized solution, the points where $u$ fails to be $C^2$ are called singularities of the solution (sometimes we abuse language and say singularities of the wave equation).

To understand what is going on, consider the case when for fixed $t_0$. $u$ is $C^2$ except at the point $(t_0,x_0)$. Writing $u(t,x) = F(x+t)+G(x-t)$, we see that $F$ is not $C^2$ at $x_0+t_0$ and/or $G$ is not $C^2$ at $x_0-t_0$. The two characteristics passing through $(t_0,x_0)$ are $x+t = x_0+t_0$ and $x-t = x_0-t_0$.

Thus, for any fixed $t_1$, $u(t_1,x)$ fails to be $C^2$ except at one or two points, namely, $x_\pm$ such that

$$x_+ + t_1 = x_0+t_0, \quad x_- - t_1 x = x_0 - t_0.$$




This shows that the *singularities of the wave equation remain localized in space and travel along the characteristics*.

We will see that the results we obtained for the 1d wave equation (existence and uniqueness for the Cauchy problem, existence of domains of influence/dependence, propagation of singularities along characteristics) hold for the wave equation in higher dimensions and, in fact, for a class of equations called hyperbolic, of which the wave equation is the prototypical example.

## 8. Some general tools, definitions, and conventions for the study of PDEs

In order to advance further our study of PDEs, in particular to study PDEs in $\mathbb{R}^n$, we will recall a few tools from multivariable calculus and introduce some convenient notation/terminology.

### 8.1. **Domains and boundaries.**

**Definition 8.1.** A domain in $\mathbb{R}^n$ is an open connected subset of $\mathbb{R}^n$. If $\Omega \subset \mathbb{R}^n$ is a domain, we denote by $\bar{\Omega}$ its closure in $\mathbb{R}^n$. The boundary of a domain $\Omega$, denoted $\partial\Omega$, is the set $\partial\Omega := \bar{\Omega} \setminus \Omega$. We say that a boundary $\partial\Omega$ has regularity $C^k$ or is a $C^k$ boundary if it can be written locally as the graph of a $C^k$ function.

**Notation 8.2.** We note by $|x|$ the Euclidean norm of an element $x \in \mathbb{R}^n$. $\Omega$ and $\partial\Omega$ will always denote a domain and its boundary, unless stated otherwise.

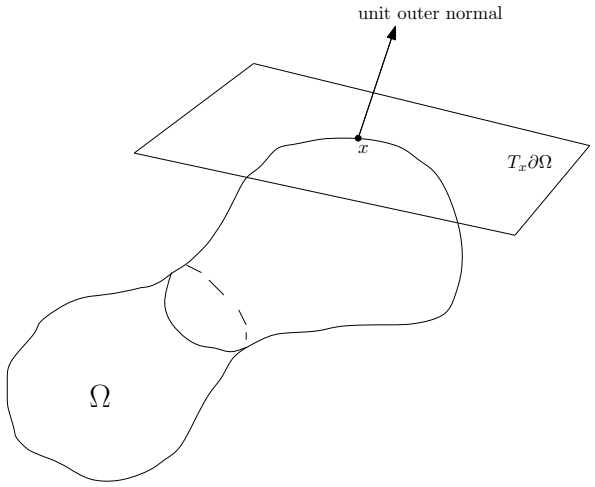
**Example 8.3.** $B^n := \{x \in \mathbb{R}^n \mid |x| < 1\}$ is a domain in $\mathbb{R}^n$. Its boundary is the $n-1$ dimensional sphere: $S^{n-1} := \partial B^n = \{x \in \mathbb{R}^n \mid |x| = 1\}$. It is not difficult to see that $S^{n-1}$ is $C^\infty$, i.e., $B^n$ has a $C^\infty$ boundary. For example, the upper cap of $S^{n-1}$, given by $S^{n-1} \cap \{x^n > 0\}$, is the graph of the function $f : B^{n-1} \subset \mathbb{R}^{n-1} \to \mathbb{R}$ given by

$$f(x^1, ..., x^{n-1}) = \sqrt{1 - (x^1)^2 - ... - (x^{n-1})^2},$$

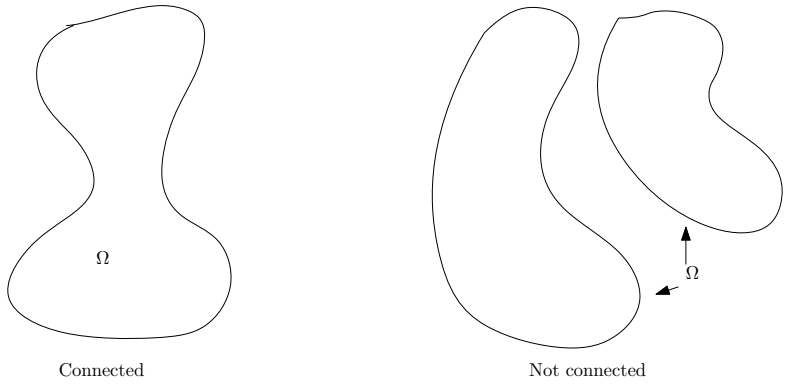
which is $C^\infty$.

**Notation 8.4.** When talking about maps between subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, we will often write $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$, where it is implicitly understood that the domain $U$ of $f$ is an open set (unless said otherwise).

Recall that if $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is $C^1$, for each $x \in U$ the graph of $f$ at $(x, f(x))$ admits a tangent plane. Thus, if $\partial\Omega$ is $C^1$, for each $x \in \partial\Omega$ there exists a tangent plan to $\partial\Omega$ at $x$, denoted $T_x\partial\Omega$. The unit outer normal to $\partial\Omega$ at $x$ is by definition the unit normal to $T_x\partial\Omega$ that points to the exterior of $\Omega$. The collection of the unit outer normals $N_x$ as $x$ varies over $\partial\Omega$ forms a vector field over $\partial\Omega$ which is called the unit outer normal vector field. We sometimes refer simply to "the unit outer normal" when the context makes it clear whether we are talking about the vector field as a specific vector field.



**Remark 8.5.** Above, we took for granted that students recall (or have seen) the definition of a connected set in $\mathbb{R}^n$. Intuitively, a set is connected if is not "split into separate parts:"




For the time being, this intuitive notion will suffice for students who have not see the precise definition. The mathematical definition of connectedness will be given later on.

8.2. **The Kronecker delta.**

**Definition 8.6.** The Kronecker delta symbol in $n$ dimensions or simply the Kronecker delta when the dimension is implicitly understood, is defined as the collection of numbers $\{\delta_{ij}\}_{i,j=1}^{n}$ such that $\delta_{i,j} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. We identify the Kronecker delta with the entries of the $n \times n$ identity matrix in the standard coordinates. We also define $\delta^{ij} := \delta_{ij}$, which we also call the Kronecker delta and identity with the entries of the identity matrix.

Recall that the Euclidean inner product, a.k.a. the dot product, of vectors in $\mathbb{R}^n$ is the map:

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

given in standard coordinates by:

$$\langle X, Y \rangle = \sum_{i=1}^{n} X^i Y^i,$$

which is also denoted by $X \cdot Y$. We can write $\langle X, Y \rangle$ as (recall over sum convention):

$$\langle X, Y \rangle = \delta_{ij} X^i Y^j$$

In view of this last formula, we also identify the Kronecker delta with the Euclidean inner product.

8.3. **Raising and lowering indices with $\delta$.** Given a vector $X = (X^1, ..., X^n)$, we define

$$X_i := \delta_{ij} X^j, \quad i = 1, ..., n.$$

We say that we are lowering the indices of $X$ and identify the $n$-tuple $(X_1, X_2, ..., X_n)$ with the vector $X$ itself.

The point of introducing $X_i$ is to achieve consistency with our convention of summing indices that appear once up and once down. For example, if we want the inner product

$$\langle X, Y \rangle = \sum_{i=1}^{n} X^i Y^i$$

using our sum convention (thus avoiding to write $\sum_{i=1}^n$), one of the indices $i$ needs to be downstairs:

$$\langle X, Y\rangle = X^i Y_i,$$

so that we had to break with our convention that vectors have indices upstairs. However, if we now interpret $Y_i$ as lowering the indices of $Y$, then

$$\langle X, Y\rangle = \delta_{ij} X^i Y^j = X^i \delta_{ij} Y^j = X^i Y_i.$$

Similarly, recall that we wrote

$$\operatorname{curl}^i X = \epsilon^{ijk} \partial_j X_k,$$

where we had artificially written $X_k$ with an index downstairs, thus breaking with our convention that vectors had an index upstairs. But now we have a proper way of thinking of $X_k$ as $\delta_{kj} X^j$.

Note that using $\delta_{ij}$ we could completely avoid writing vectors with indices downstairs, i.e., every time that $X_i$ appears in a forumla we can replace it with $\delta_{ij} X^j$. E.g.,

$$\operatorname{curl}^i X = \epsilon^{ijk} \delta_{k\ell} \partial_j X^\ell.$$

But the point is precisely to have a compact notation, so $\delta_{k\ell} \partial X^\ell = \partial_j \delta_{k\ell} X^\ell = \partial_j X_k$.

**Remark 8.7.** In the above computations, note that we can move $\delta_{k\ell}$ pass the derivative because $\delta_{k\ell}$ is constant for each fixed $k$ and $\ell$, i.e., $\delta_{k\ell}$ is not a function of the coordinates.

We extend the lowering of indices to any object indexed by $i_1, ..., i_\ell$, $i_j \in \{1, ..., n\}$, $j = 1, ..., n$. E.g.:

$$\epsilon_i{}^{jk} := \delta_{i\ell} \epsilon^{\ell jk},$$
$$\epsilon^i{}_j{}^k := \delta_{j\ell} \epsilon^{i\ell k}, \text{ etc.}$$

Note that it is important to keep the order of the indices on the LHS due to the anti-symmetry of $\epsilon$, so that $\epsilon^i{}_j{}^k \neq \epsilon_i{}^{jk}$. In fact, the order of indices always matter unless one is dealing with objects that are symmetric in the respective indices. E.g., if $a^{ij}$ are the entires of a matrix, then

$$a_i{}^j := \delta_{i\ell} a^{\ell j}$$

and in general $a_i{}^j \neq a^j{}_i$. However, if the matrix is symmetric, $a^{ij} = a^{ji}$, then $a_i{}^j = a^j{}_i$, and we write $a_i^j$ for $a_i{}^j$.

The same way we lowered indices using $\delta_{ij}$, we can raise indices using $\delta^{ij}$. For instance, given an object indexed by downstairs indices $ij$, i.e., $A_{ij}$, we set

$$A^i{}_j := \delta^{i\ell} A_{\ell j}.$$

Again, the order of the indices on the LHS matters unless the object is symmetric. It follows that we can define the Kronecker delta with one index up and one down:

$$\delta^i_j = \delta^{i\ell} \delta_{\ell j}.$$

It follows that

$$\delta^i_j = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Note that raising and then lowering (or vice-versa) an index gives the same object back. E.g.:

$$X_i = \delta_{ij} X^j \Rightarrow X^i = \delta^{ij} X_j = \underbrace{\delta^{ij}\delta_{j\ell}}_{\delta^i_\ell} X^\ell = X^i,$$

where we used $\delta^i_\ell = 0$ for $i \neq \ell$.

Recall that $\partial_i = \frac{\partial}{\partial x^i}$. We define the derivative with an index upstairs by:

$$\partial^i := \delta^{ij}\partial_j.$$

Using this notation, we can write the Laplacian as:

$$\Delta = \partial^i \partial_i = \delta^{ij}\partial_i\partial_j.$$

We sometimes abbreviate $\partial^2_{ij} = \partial_i\partial_j, \partial^3_{ijk} = \partial_i\partial_j\partial_k$, etc.

**Important remark.** The use of the Kronecker delta and the raising and lowering of indices provide us with a convenient and compact notation. But the overall discussion and definitions probably seem a bit ad hoc. It turns out that these ideas can be given a more satisfactory content within the language of differential geometry. For example, the Kronecker delta can be introduced not as a "collection of symbols" but rather as a tensor satisfying certain properties. The raising the lowering of indices can be interpreted as a map, given by the inner product, that identifies elements of a vector space and its dual, on vector fields and one forms; or yet more generally as the identification of covariant and contravariant tensors. Since we will not be discussing differential geometry (except for some elementary aspects tied to PDEs), here we will take a purely instrumental point of view, using the above machinery mostly as a matter of convenient notation.

8.4. **Calculus facts.** We collect a few calculus facts that we will use alter on.

**Definition 8.8.** We say that a map $f$ is $k$-times continuously differentiable if all its partial derivatives up to order $k$ exist and are continuous in the domain of $f$. We denote the space of $k$-times continuously differentiable functions in $U \subset \mathbb{R}^n$ by $C^k(U)$. Sometimes we write simply $C^k$ if $U$ is implicitly understood, and sometimes we say simply "$f$ is $C^k$" to mean that $f$ is $k$-times continuously differentiable.

**Integration by parts.** If $u, v \in C^1(\bar{\Omega})$, then

$$\int_\Omega \partial_i uv \mathrm{d}x = -\int_\Omega u\partial_i v \mathrm{d}x + \int_{\partial\Omega} uv\nu^i \mathrm{d}S,$$

$i = 1, ...n$, where $\nu = (\nu^1, ..., \nu^n)$ is the unit outer normal to $\partial\Omega$ and $\mathrm{d}S$ is the volume element induced on $\partial\Omega$.

Students who have not seen the above integration by parts in $\mathbb{R}^n$ can view it as a generalization of the divergence theorem in $\mathbb{R}^3$. The latter can be written (using Stewart's Calculus notation):

$$\iiint_E \operatorname{div} \overrightarrow{F} \mathrm{d}V = \iint_S \overrightarrow{F} \cdot \mathrm{d}\overrightarrow{S}.$$

Take $\overrightarrow{F} = uv\overrightarrow{e}_i$, where $\overrightarrow{e}_i$ has 1 in the $i^{th}$ component and zero in the remaining components. Then,

$$\operatorname{div} \overrightarrow{F} = \partial_i uv + u\partial_i v.$$

For example, if $\overrightarrow{e}_i = e_1 = (1,0,0)$, and writing $\overrightarrow{F} = (F_x, F_y, F_z)$, so that

$$\operatorname{div} \overrightarrow{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z,$$

we find

$$\operatorname{div} \overrightarrow{F} = \operatorname{div}(uv, 0, 0) = \partial_x(uv) = \partial_x uv + u\partial_x v,$$

and similarly for $\overrightarrow{e}_2$ and $\overrightarrow{e}_3$. Recalling also that $\mathrm{d}\overrightarrow{S} = \overrightarrow{n}\mathrm{d}S$, where $\overrightarrow{n}$ is the unit outer normal,

$$\overrightarrow{F} \cdot \mathrm{d}\overrightarrow{S} = (uv\overrightarrow{e}_i) \cdot \overrightarrow{n}\mathrm{d}S = uv\overrightarrow{e}_i \cdot \overrightarrow{n}\mathrm{d}S.$$

But $\overrightarrow{e}_i \cdot \overrightarrow{n} = i^{th}$ component of $\overrightarrow{n} = n^i$, thus

$$\overrightarrow{F} \cdot \mathrm{d}\overrightarrow{S} = uvn^i.$$

Plugging the above into the divergence theorem:

$$\iiint_E (\partial_i uv + u\partial_i v)\mathrm{d}V = \iint_S uvn^i \mathrm{d}S$$

which is the formula we stated in a different notation.

**Definition 8.9.** Let $u \in C^1(\bar{\Omega})$. The normal derivative of $u$, denoted $\frac{\partial u}{\partial \nu}$, is a function defined on $\partial\Omega$by

$$\frac{\partial u}{\partial \nu} := \nabla u \cdot \nu,$$

where $\nu$ is the unit outer normal to $\partial\Omega$ and $\nabla$ is the gradient.

From the integration by parts formula we can derive the following formulas (sometimes called Green's identities): for $u \in C^1(\bar{\Omega})$:

$$\int_\Omega \partial_i u \mathrm{d}x = \int_{\partial\Omega} u\nu^i \mathrm{d}S.$$

For $u, v \in C^2(\bar{\Omega})$:

$$\int_\Omega \Delta u \mathrm{d}x = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\mathrm{d}S,$$

$$\int_\Omega \nabla u \cdot \nabla v \mathrm{d}x = -\int_\Omega u\Delta v \mathrm{d}x + \int_{\partial\Omega} u\frac{\partial v}{\partial \nu}\mathrm{d}S,$$

$$\int_\Omega (u\Delta v - v\Delta u)\mathrm{d}x = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right)\mathrm{d}S.$$

## 9. Formal aspects of PDEs

**Definition 9.1.** A vector of the form

$$\alpha = (\alpha_1, ..., \alpha_n),$$

where each entry is a non-negative integer is called a multiindex of order $|\alpha| = \alpha_1 + ... + \alpha_n$. Given a multiindex, we define:

$$D^\alpha u := \frac{\partial^{|\alpha|} u}{\partial (x^1)^{\alpha_1} ... \partial (x^n)^{\alpha_n}},$$

where $u = u(x^1, ..., x^n)$. If $k$ is a non-negative integer,

$$D^k u := \{D^\alpha u \mid |\alpha| = k\}$$

is the set of all $k$-th order partial derivatives of $u$. When $k = 1$ we identify $Du$ with the gradient of $u$. When $k = 2$ we identify $D^2u$ with the Hessian matrix of $u$:

$$D^2u = \begin{bmatrix} \frac{\partial^2 u}{\partial (x^1)^2} & \cdots & \frac{\partial^2 u}{\partial x^1 \partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 u}{\partial x^n \partial x^1} & \cdots & \frac{\partial^2 u}{\partial (x^n)^2} \end{bmatrix}$$

We can regard $D^k u(x)$ as a point in $\mathbb{R}^{n^k}$. Its norm is

$$|D^k u(x)| = \sqrt{\sum_{|\alpha|=k} |D^\alpha u(x)|^2},$$

where $\sum_{|\alpha|\leq k}$ means the sum is over all multiindices of order $k$. If $u = (u^1, ..., u^m)$ is vector valued, we define

$$D^\alpha u := (D^\alpha u^1, ..., D^\alpha u^m)$$

and set

$$D^k u := \{D^\alpha u \mid |\alpha| = k\},$$

and

$$|D^k u| = \sqrt{\sum_{|\alpha|=k} |D^\alpha u(x)|^2}$$

as before.

We will now restate the definition of PDEs using the above notation. This new definition agrees with the one previously given.

**Definition 9.2.** Let $\Omega \subset \mathbb{R}^n$ be a domain and $k \geq 1$ be a non-negative integer. An expression of the form

$$F(D^k u(x), D^{k-1} u(x), ..., Du(x), u(x), x) = 0,$$

$x \in \Omega$, is called a $k$-th order partial differential equation (PDE), where:

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times ... \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$$

is given and $u : \Omega \to \mathbb{R}$ is unknown. A solution to the PDE is a function $u$ that verifies the PDE. Sometimes we drop $x$ from the notation and state the PDE as

$$F(D^k u, D^{k-1} u, ..., Du, u, x) = 0 \text{ in } \Omega.$$

$\Omega$ is sometimes called the domain of definition of the PDE.

**Example 9.3.** $\Delta u = 0$ in $\mathbb{R}^3$ can be written as

$$F(D^2 u, Du, u, x) = 0 \text{ in } \mathbb{R}^3$$

with $F : \mathbb{R}^9 \times \mathbb{R}^3 \times \mathbb{R} \times \underbrace{\mathbb{R}^3}_{\Omega} \to \mathbb{R}$ given by the following expression. First, we label the coordinates in $\mathbb{R}^9 \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3$ according to the order of the derivatives and $x$, i.e.,

$$\frac{\partial^2 u}{\partial (x^1)^2}, \frac{\partial^2 u}{\partial x^1 \partial x^2}, ..., \frac{\partial^2 u}{\partial (x^3)^2}, \frac{\partial u}{\partial x^1}, ..., \frac{\partial u}{\partial x^3}, u, x^1, x^2, x^3,$$

so

$$F = F(\underbrace{p_{11}, p_{12}, p_{13}, p_{21}, ..., p_{23}}_{9 \text{ entries}}, \underbrace{p_1, p_2, p_3}_{3 \text{ entries}}, p, x^1, x^2, x^3).$$

Then $F$ is given by

$$F(p_{11}, ..., x^3) = p_{11} + p_{22} + p_{33}.$$

**Example 9.4.** $\Delta u = f$ in $\mathbb{R}^3$, where $f(x) = (x^1)^2 + (x^2)^2 + (x^3)^2$, can be written, using the notation of the previous example, as in the definition with $F$ given by

$$F(p_{11}, ..., x^3) = p_{11} + p_{22} + p_{33} - ((x^1)^2 + (x^2)^2 + (x^3)^2).$$

**Definition 9.5.** A PDE

$$F(D^k u, D^{k-1}u, ..., Du, u, x) = 0$$

is called linear if $F$ is linear in all its entries except possibly in $x$. Otherwise it is called non-linear. More precisely, denoting $F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times ... \times \mathbb{R}^n \times \mathbb{R}^n \Omega \to \mathbb{R}$, by $F = F(\overrightarrow{p}, x)$

$$\overrightarrow{p} = (\underbrace{p_{k,1}, ..., p_{k,n^k}}_{n^k \text{ entries for } \mathbb{R}^{n^k}}, \underbrace{p_{k-1,1}, ..., p_{k-1,n^{k-1}}}_{n^{k-1} \text{ entries for } \mathbb{R}^{n^{k-1}}}, ..., p),$$

we can write $F(\overrightarrow{p}, x) = F_H(\overrightarrow{p}, x) + F_I(x)$, where $F_I$ contains all terms that do not depend on $\overrightarrow{p}$ (i.e., terms that do not depend on $u$ or its derivatives). $F_H$ is called the homogeneous part of $F$ and $F_I$ the inhomogeneous part. Thus, the PDE is called linear if $F_H$ is linear in all its entries. The PDE is called homogenous if $F_I = 0$ and inhomogeneous otherwise.

We clarify that when we say that $F_H$ is linear in, say, the entry $D^k u$, we mean that it is linear in each component of $D^k u$ separately. For instance, $F_H(Du, u, x) = 0$ is linear if it is linear in $Du$. Since $Du = (\partial_1 u, ..., \partial_n u)$ we mean that $F$ is linear in each entry of $(\partial_1 u, ..., \partial_n u)$ plus in the entry $u$. In other words, labeling the entries of $F = F(p_1, ..., p_n, p, x^1, ..., x^n)$, the PDE is linear if $F$ is linear in each $p_i$, $i = 1, 2, ..., n$, and in $p$. A linear PDE $F(D^k u, ..., u, x)$ can always be written as

$$\sum_{|\alpha| \le k} a_\alpha D^\alpha u = f,$$

where the $a_\alpha$ and $f$ are known functions defined on $\Omega$. If the PDE is also homogenous then $f = 0$. A PDE as defined above, where the unknown is a single function on $\Omega$, is also called a scalar PDE.

**Definition 9.6.**

(1) A $k^{th}$ order PDE is called semi-linear if it has the form

$$\sum_{|\alpha| = k} a_\alpha D^\alpha u + a_0(D^{k-1}u, .., Du, u, x) = 0,$$

where the $a_\alpha : \Omega \to \mathbb{R}$ and $a_0 : \mathbb{R}^{n^{k-1}} \times ... \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ are given functions.

(2) A $k^{th}$ order PDE is called quasi-linear if it has the form

$$\sum_{|\alpha| = k} a_\alpha(D^{k-1}u, .., Du, u, x) D^\alpha u + a_0(D^{k-1}u, .., Du, u, x) = 0,$$

where $a_\alpha, a_0 : \mathbb{R}^{n^{k-1}} \times ... \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ are known.

(3) A PDE is called fully nonlinear if it depends nonlinearly on its highest order derivative.

**Definition 9.7.** An expression of the form

$$F(D^k u(x), D^{k-1}u(x), ..., Du(x), u(x), x) = 0,$$

is called a $k^{th}$ order system of PDEs, where

$$F = (F^1, ..., F^\ell) : \mathbb{R}^{mn^k} \times \mathbb{R}^{mn^{k-1}} \times ... \times \mathbb{R}^{mn} \times \mathbb{R}^m \times \Omega \to \mathbb{R}^\ell$$

is given and $u = (u^1, ..., u^m) : \Omega \to \mathbb{R}^m$ is the unknown. A solution to the system of PDEs is a function $u : \Omega \to \mathbb{R}^m$ that satisfies the system of PDEs. We sometimes drop the $x$-dependence and write

$$F(D^k u, ..., Du, u, x) = 0$$

in $\Omega$. We sometimes refer to a system of PDEs simply as a PDE.

The definitions of (non)linear, (non)homogenous, semi-linear, quasi-linear generalize in a straightforward fashion to systems. In particular, a linear system can be written as

$$\sum_{|\alpha| \le k} A_\alpha D^\alpha u = f,$$

where $A_\alpha : \Omega \to \mathbb{R}^{\ell m}$ are known $\ell \times m$ matrices (depending on $x \in \Omega$) and $f : \Omega \to \mathbb{R}^\ell$ is a known function ($f = 0$ if the system is homogeneous).

Having introduced the basic definitions and terminology for PDEs, let us discuss the case of evolution equations, i.e., when one of the variables represents time.

When we study a PDE where one of the variables is the time variable, it is convenient to separate time and space and denote the spatial variables by $(x^1, ..., x^n)$ and the time variables by $x^0$. In this case we have $n + 1$ variables and extend the multi-index notation to

$$\alpha = (\alpha_0, ..., \alpha_n), \quad |\alpha| = \alpha_0 + ... + \alpha_n, \quad D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial (x^0)^{\alpha_0} \partial (x^1)^{\alpha_1} ... \partial (x^n)^{\alpha_n}}.$$

The domain of definition of the PDE in this case is $\Omega \subset \mathbb{R}^{n+1}$, but it is convenient to take it to be $(T_I, T_F) \times \Omega \subset \mathbb{R}^{n+1}$, for some interval $(T_I, T_F) \subset \mathbb{R}$ and some domain $\Omega \subset \mathbb{R}^n$. Typically $(T_I, T_F) = (0, T)$ for some $T > 0$. We also write $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$ wen we want to emphasize the the first coordinate, $x^0$, correspond to time. We also write $t := x^0$ for the time variable. Thus, $\frac{\partial}{\partial t} = \frac{\partial}{\partial x^0}$.

**Notation 9.8.** We extend our indices convention by adopting the convention that Latin Lower-case indices range from 1 to $n$ (as we have used so far) and Greek lower-case indices range from 0 to $n$. For instance,

$$\begin{aligned} a^\alpha \partial_\alpha u &= a^0 \partial_0 u + a^i \partial_i u \\ &= a^0 \partial_t u + a^i \partial_i u \\ &= a^0 \partial_t u + a^1 \partial_1 u + ... + a^n \partial_n u. \end{aligned}$$

Note that we use Greek letters to denote both indices varying from 0 to $n$ and multi-indices. The context will make the distinction clear. In particular, note that for multi-indices we never use the convention that repeated indices are summed. Thus, for example, in $a^\alpha \partial_\alpha$, $\alpha$ is an index summed from 0 to $n$, whereas in $\sum_{|\alpha| \le k} a^\alpha D^\alpha$, $\alpha$ is a multi-index summed over all multi-indices with $|\alpha| \le k$. Finally, if $\alpha = (\alpha_0, ..., \alpha_n)$ is a multi-index, we write $\overrightarrow{\alpha}$ for its "spatial part," i.e., $\overrightarrow{\alpha} = (\alpha_1, ..., \alpha_n)$.

We next state some useful calculus facts using multi-index notation. The formulas below involve functions $u = u(x^1, ..., x^n)$ and $\alpha = (\alpha_1, ..., \alpha_n)$, but clearly similar formulas hold for $u = u(x^0, x^1, ..., x^n)$ and $\alpha = (\alpha_0, \alpha_1, ..., \alpha_n)$. For multi-indices $\alpha$ and $\beta$ define

$$\alpha! = \alpha_1!\alpha_2!...\alpha_n!, \qquad \alpha \leq \beta \Leftrightarrow \alpha_i \leq \beta_i \text{ for each } i = 1, ..., n, \qquad x^\alpha = \prod_{i=1}^{n} x_i^{\alpha_i}.$$

(1) **Multinomial theorem**:

$$(x_1 + ... + x_n)^k = \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} x^\alpha, \text{ where } \binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha!}$$

(2) **Leibniz's formula or product rule**:

$$D^\alpha(uv) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta u D^{\alpha-\beta} v, \text{ where } \binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}.$$

(3) **Taylor's forumla**:

$$u(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha u(0) x^\alpha + \mathcal{O}(|x|^{k+1}) \text{ as } x \to 0.$$

Above, $u, v : \mathbb{R}^n \to \mathbb{R}$ are sufficiently regular as to make the formulas valid.

**Remark 9.9.** When we introduce a PDE, we indicate the domain $\Omega$ where it is defined, which says that we are looking for a solution that is defined in $\Omega$. It may happen, however (and it is often the case for non-linear PDEs) that we are able to find a solution $u$ but $u$ is defined only on a smaller domain $\Omega' \subset \Omega$. I.e., $u$ satisfies the PDE only for $x \in \Omega'$, where $\Omega'$ is strictly smaller than $\Omega$. In fact, we a priori do not know whether it is possible to satisfy the PDE for all $x \in \Omega$. We still call such a $u$ that is defined only on $\Omega'$ a solution, and sometimes call it a local solution if we want to emphasize that the solution we found is defined on a domain smaller than where the PDE was originally stated. In other words, the domain of definition of the PDE is a guide that helps us define the problem, but it can happen that solution are only defined in a subset of $\Omega$.

**Example 9.10.** Let us illustrate this situation with a simple ODE example. Consider the Riccati equation $y' = y^2$ in $\Omega = (0, \infty)$ with initial condition $y(0) = 1$. The solution is $y(t) = \frac{1}{1-t}$. This solution, however, is not defined for $t = 1$. Thus we in fact have a local solution defined on $\Omega = (0, 1)$ (we do not take $\Omega' = (0, 1) \cup (1, \infty)$ because this set is not connected; and we take the portion (0,1)because we need to approached zero to satisfy the initial condition)

We can also defined boundary value problems, initial value problems, and initial-boundary value problems as we had done for the 1d wave equation. We will not give these general definitions here, but will introduce them as needed to study specific problems. We note that in such cases we will in general seek a solution defined on a larger domain than $\Omega$. For example, we may want $u : \bar{\Omega} \to \mathbb{R}$ in a boundary value problem. What exactly is required is usually a case-by-case analysis.

**Important notation on constants** In what follows we are going to derive estimates and computations that involve numerical constants whose specific value will not be important. Thus, we will denote by $C > 0$ a generic positive constant that can vary from line to line.

$C$ will generally depend on a fixed data of the problem (e.g., the dimension $n$). Sometimes we indicate the dependence of $C$ using subscripts or function notation, e.g. $C_n$ or $C(n)$.

## 10. Laplace's Equation in $\mathbb{R}^n$

We are going to study Laplace's equation in $\mathbb{R}^n$:

$$\triangle u = 0 \text{ in } \mathbb{R}^n,$$

and its inhomogeneous version known as Poisson's equation:

$$\triangle u = f \text{ in } \mathbb{R}^n,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is given.

We begin looking for a solution of the form $u(x) = v(r)$ where $r = |x| = ((x^1)^2 + ... + (x^n)^2)^{1/2}$ is the distance to the origin. The motivation to look for such a solution is that Laplace's equation is rotationally invariant (this will be a HW). Direct computation gives:

$$\partial_i r = \frac{x^i}{r}, x \neq 0, \quad \partial_i u = v' \frac{x^i}{r}, \quad \partial_i^2 u = v'' \frac{(x^i)^2}{r^2} + v' \left( \frac{1}{r} - \frac{(x^i)^2}{r^3} \right).$$

Summing from 1 to $n$:

$$\triangle u = v'' + \frac{n-1}{r} v'.$$

Hence $\triangle u = 0$ iff $v'' + \frac{n-1}{r} v' = 0$, which is an ODE for $v$. If $n' \neq 0$ we cam write it as

$$(\ln(v'))' = \frac{v''}{v'} = \frac{1-n}{r},$$

which gives $v'(r) = \frac{A}{r^{n-1}}$ for some constant $A$. If $r > 0$, integrating again we find

$$v(r) = \begin{cases} a \ln r + b, & n = 2 \\ \frac{a}{r^{n-2}} + b, & n \geq 3, \end{cases}$$

where $a$ and $b$ are arbitrary constants. This calculation motivates the following definition.

**Definition 10.1.** The function

$$\Gamma(x) := \begin{cases} \frac{1}{2\pi} \ln |x|, & n = 2 \\ \frac{1}{n(2-n)\omega_n} \frac{1}{|x|^{n-2}}, & n \geq 3, \end{cases}$$

is called the fundamental solution of Laplace's equation.

**Notation 10.2.** We denote by $B_r(x)$ the (open) ball of radius $r$ centered at $x$ in $\mathbb{R}^n$, i.e.,

$$B_r(x) := \{ y \in \mathbb{R}^n \mid |x - y| < r \}.$$

Sometimes we write $B_r^n(x)$ to emphasize the dimension. We denote

$$\omega_n := \text{vol}(B_1^n(0)).$$

In particular, $\omega_3 = \frac{4}{3}\pi$.

Note that $\triangle \Gamma(x) = 0$ for $x \neq 0$ by construction. Sometimes we write $\Gamma(|x|)$ to emphasize the radial dependence on $r = |x|$. Before solving Laplace's equation, we need one more definition.

**Definition 10.3.** The support of a map $f : U \to \mathbb{R}$ is the set

$$\text{supp}(f) := \overline{\{x \in U \mid f(x) \neq 0\}},$$

where the bar is the closure. Recall that a set $U \subset \mathbb{R}^n$ is called compact if it is closed and bounded. We say that $f$ has compact support if $\text{supp}(f)$ is compact. We denote by $C_c^k(U)$ the space of $C^k$ functions in $U$ with compact support.

**Theorem 10.4.** *Let $f \in C_c^2(\mathbb{R}^n)$. Set*

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y)f(y)\mathrm{d}y = (\Gamma * f)(x).$$

*Then:*

*(1) $u$ is well-defined*
*(2) $u \in C^2(\mathbb{R}^n)$*
*(3) $\triangle u = f$ in $\mathbb{R}^n$.*

*Proof.* We will carry out the proof for $n \geq 3$. The case $n = 2$ is done with similar arguments.

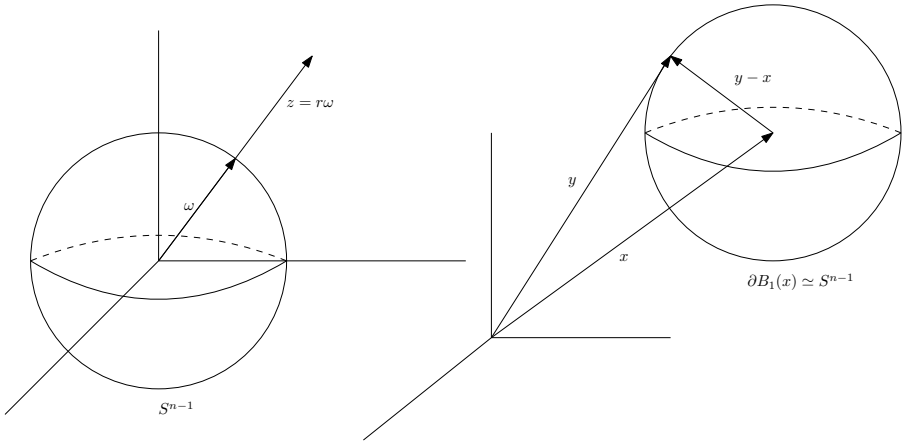
To begin, recall that a continuous function over a compact set is always has a maximum and minimum. Therefore, since $f$ has compact support, there exists a constant $C > 0$ such that $|f(x)| \leq C$ for every $x$. Moreover, again by the compact support of $f$, there exists a $R > 0$ such that

$$\int_{\mathbb{R}^n} \Gamma(x-y)f(y)\mathrm{d}y = \int_{B_R(x)} \Gamma(x-y)f(y)\mathrm{d}y.$$

Thus,

$$\left| \int_{\mathbb{R}^n} \Gamma(x-y)f(y)\mathrm{d}y \right| \leq C \int_{B_R(x)} |\Gamma(x-y)|\mathrm{d}y \leq C \int_{B_R(x)} \frac{1}{|x-y|^{n-2}}\mathrm{d}y.$$

We now take polar coordinates $(r, \omega)$ centered at $x$, where $r =$ distance to $x$ and $\omega \in S^{n-1}$ ($n-1$ dimensional unit sphere), sothat $y - x = r\omega$, $|x - y| = r$.




In these coordinates $\mathrm{d}y = r^{n-1}\mathrm{d}\omega$, where $\mathrm{d}\omega$ is the volume element on $S^{n-1}$ ( for $n = 3, \mathrm{d}\omega = \sin(\phi)\mathrm{d}\phi\mathrm{d}\theta$). Then

$$\int_{B_R(x)} \frac{1}{|x-y|^{n-2}}\mathrm{d}y = \int_0^R \int_{S^{n-1}} \frac{1}{r^{n-2}} r^{n-1}\mathrm{d}r\mathrm{d}\omega = \int_0^R r\mathrm{d}r \int_{S^{n-1}} \mathrm{d}\omega = C,$$

showing that $u$ is well defined, i.e., (1).

To prove (2), first make a change of variables $z = x - y$, so

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y)f(y)\mathrm{d}y = \int_{\mathbb{R}^n} \Gamma(z)f(x-z)\mathrm{d}z.$$

Note that $\partial_i f$ and $\partial^2_{ij} f$ also have compact support, thus an argument to the above shows that

$$\int_{\mathbb{R}^n} |\Gamma(y)\partial_i f(x-y)|\mathrm{d}y \text{ and } \int_{\mathbb{R}^n} |\Gamma(y)\partial^2_{ij} f(x-y)|\mathrm{d}y$$

are well defined. Let $e_i = (0, ..., 1, ..., 0)$ be the canonical basis vectors in $\mathbb{R}^n$ and let $h > 0$. Then, for any $x$:

$$\begin{aligned}\frac{u(x+he_i)-u(x)}{h} &= \int_{\mathbb{R}^n} \Gamma(y)\left(\frac{f(x+he_i-y)-f(x-y)}{h}\right)\mathrm{d}y \\ &= \int_{B_R(0)} \Gamma(y)\left(\frac{f(x+he_i-y)-f(x-y)}{h}\right)\mathrm{d}y,\end{aligned}$$

where the second equality holds for a sufficiently large $R$ in view of the compact support of $f$. Since

$$\lim_{h\to 0} \frac{f(x+e_ih-y)-f(x-y)}{h} = \partial_i f(x-y)$$

and the integral of $\Gamma(y)\partial_i f(x-y)$ is well defined,

$$\begin{aligned}\lim_{h\to 0}\frac{u(x+he_i)-u(x)}{h} &= \lim_{h\to 0}\int_{\mathbb{R}^n} \Gamma(y)\left(\frac{f(x+he_i-y)-f(x-y)}{h}\right)\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \Gamma(y)\left(\lim_{h\to 0}\frac{f(x+he_i-y)-f(x-y)}{h}\right)\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \Gamma(y)\partial_i f(x-y)\mathrm{d}y,\end{aligned}$$

showing that the limit $\lim_{h\to 0}\frac{u(x+he_i)-u(x)}{h}$ exists, i.e., $\partial_i u(x)$ exists. Repeating this argument with $f(x-y)$ replaced by $\partial_i f(x-y)$ we conclude that $\partial^2_{ij}u(x)$ exists and

$$\partial^2_{ij}u(x) = \int_{\mathbb{R}^n} \Gamma(y)\partial^2_{ij} f(x-y)\mathrm{d}y.$$

To show that $u \in C^2(\mathbb{R}^n)$, it remains to show that $\partial^2_{ij}u$ is continuous. Fix $x_0 \in \mathbb{R}^n$ and $\epsilon > 0$, and consider:

$$\begin{aligned}|\partial^2_{ij}u(x_0) - \partial^2_{ij}u(x)| &= \left|\int_{\mathbb{R}^n} \Gamma(y)(\partial^2_{ij}f(x_0-y) - \partial^2_{ij}f(x-y))\mathrm{d}y\right| \\ &\le \int_{\mathbb{R}^n} |\Gamma(y)||\partial^2_{ij}f(x_0-y) - \partial^2_{ij}f(x-y)|\mathrm{d}y.\end{aligned}$$

Since $\partial^2_{ij}f$ is continuous and has compact support it is uniformly continuous, i.e., given $\epsilon'$, there exists a $\delta > 0$ such that $|\partial^2_{ij}f(z) - \partial^2_{ij}f(y)| < \epsilon'$ whenever $|z-y| < \delta$. Putting $\epsilon' = \frac{\epsilon}{C}$, with $C = \int_{B_R(0)} |\Gamma(y)|\mathrm{d}y$ ( which we already know to be finite), we find that if $|x_0 - x| < \delta$, so that $|(x_0-y)-(x-y)| < \delta$, we obtain that

$$|\partial^2_{ij}u(x_0) - \partial^2_{ij}u(x)| \le \int_{B_R(0)} |\Gamma(y)| \underbrace{|\partial^2_{ij}f(x_0-y) - \partial^2_{ij}f(x-y)|}_{<\epsilon'}\mathrm{d}y < \epsilon,$$

showing that $u \in C^2(\mathbb{R}^n)$.

To show (3), from the expression for $\partial_{ij}u$ we obtain

$$\begin{aligned}\triangle u(x) = \delta^{ij}\partial^2_{ij}u(x) &= \int_{\mathbb{R}^n} \Gamma(y)\triangle_x f(x-y)\mathrm{d}y,\\ &= \int_{\mathbb{R}^n\setminus B_\epsilon(0)} \Gamma(y)\triangle_x f(x-y)\mathrm{d}y + \int_{B_\epsilon(0)} \Gamma(y)\triangle_x f(x-y)\mathrm{d}y =: I_1^\epsilon + I_2^\epsilon,\end{aligned}$$

where $\epsilon > 0$ and we write $\triangle_x$ to emphasize that in $\triangle_x f(x-y)$ the Laplacian is with respect to the $x$ variable. Noticing that $\triangle_x f(x-y) = \triangle_y f(x-y)$, Green's identities give:

$$\begin{aligned}I_1 &= \int_{\mathbb{R}^n\setminus B_\epsilon(0)} \Gamma(y)\triangle_y f(x-y)\mathrm{d}y\\ &= 0\int_{\mathbb{R}^n\setminus B_\epsilon(0)} \nabla\Gamma(y)\cdot\nabla_y f(x-y)\mathrm{d}y + \int_{\partial B_\epsilon(0)} \Gamma(y)\frac{\partial f}{\partial\nu}(x-y)\mathrm{d}S(y)\\ &=: I_{11}^\epsilon + I_{12}^\epsilon,\end{aligned}$$

where we write $\nabla_y$ and $\mathrm{d}S(y)$ to emphasize that the gradient and integration over $\partial B_\epsilon(0)$ are on the $y$ variable. We also notice that in the integration by parts there is no term to be "evaluated at $\infty$" since $f$ has compact support.

Let's now analyze the integrals $I_2^\epsilon, I_{11}^\epsilon$, and $I_{12}^\epsilon$. Observe that:

$$\begin{aligned}|I_2^\epsilon| &\le \int_{B_\epsilon(0)} |\Gamma(y)|\underbrace{|\triangle_x f(x-y)|}_{C} \le C\int_{B_\epsilon(0)} |\Gamma(y)|\mathrm{d}y\\ &\le C\int_0^\epsilon \frac{1}{r^{n-2}} r^{n-1}\mathrm{d}r = C\epsilon^2.\end{aligned}$$

Since $\mathrm{d}S(y) = \epsilon^{n-1}\mathrm{d}\omega$ and $|\Gamma(y)| \le C/\epsilon^{n-2}$ on $\partial B_\epsilon(0)$:

$$|I_{12}^\epsilon| \le \int_{\partial B_\epsilon(0)} |\Gamma(y)|\left|\frac{\partial f}{\partial\nu}(x-y)\right|\mathrm{d}S(y) \le C\epsilon.$$

For $I_{11}^\epsilon$, we integrate by parts again:

$$\begin{aligned}I_{11}^\epsilon &= -\int_{\mathbb{R}^n\setminus B_\epsilon(0)} \nabla\Gamma(y)\cdot\nabla_y f(x-y)\mathrm{d}y\\ &= \int_{\mathbb{R}^n\setminus B_\epsilon(0)} \triangle\Gamma(y) f(x-y)\mathrm{d}y - \int_{\partial B_\epsilon(0)} \frac{\partial\Gamma}{\partial\nu}(y) f(x-y)\mathrm{d}S(y)\\ &= 0 - \int_{\partial B_\epsilon(0)} \frac{\partial\Gamma}{\partial\nu}(y) f(x-y)\mathrm{d}S(y),\end{aligned}$$

where we used that $\triangle\Gamma(y) = 0$ for $y \neq 0$.

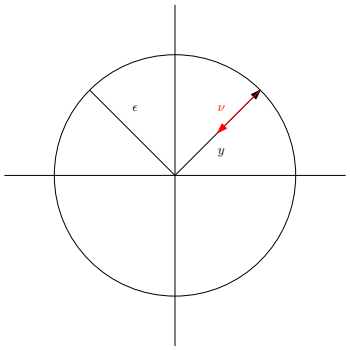
From the explicit expression for $\Gamma(y)$, compute:

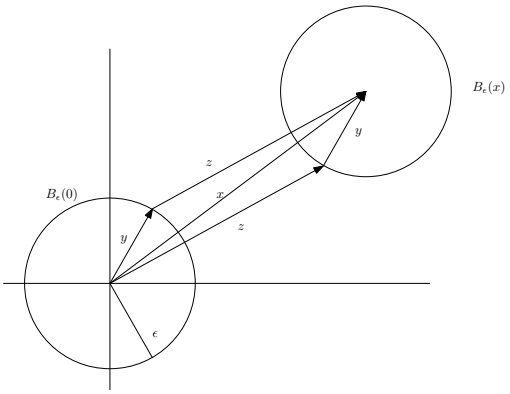
$$\nabla\Gamma(y) = \frac{1}{n\omega_n}\frac{y}{|y|^n}, \quad y \neq 0.$$

The unit outer normal in the integral is given by $\nu = \frac{-y}{|y|}$, thus

$$I_{11}^\epsilon = \int_{\partial B_\epsilon(0)} \frac{1}{n\omega_n}\frac{|y|^2}{|y|^{n+1}} f(x-y)\mathrm{d}S(y) = \frac{1}{n\omega_n\epsilon^{n-1}}\int_{\partial B_\epsilon(0)} f(x-y)\mathrm{d}S(y),$$

since $|y| = \epsilon$ on $\partial B_\epsilon(0)$.




Making a change of variables $x - y = z$, we find

$$I_{11}^\epsilon = \frac{1}{n\omega_n \epsilon^{n-1}} \int_{\partial B_\epsilon(0)} f(z) \mathrm{d}S(z).$$

Note that $n\omega_n\epsilon^{n-1}$ is the surface are, or volume, of $\partial B_\epsilon(0)$ (e.g., for $n = 3, n\omega_n\epsilon^{n-1} = 4\pi\epsilon^2$), so we write

$$I_{11}^\epsilon = \frac{1}{\mathrm{vol}(\partial B_\epsilon(0))} \int_{\partial B_\epsilon(x)} f(y) \mathrm{d}S(y).$$

Since we have

$$\triangle u(x) = \int_{\mathbb{R}^n \setminus B_\epsilon(0)} (\dots) + \int_{B_\epsilon(0)} (\dots) = I_1^\epsilon + I_2^\epsilon$$

which is valid for any $\epsilon > 0$, we conclude that

$$\triangle u(x) = \lim_{\epsilon \to 0^+} I_1^\epsilon + \lim_{\epsilon \to 0^+} I_2^\epsilon$$

if the limits exist. From the foregoing:

$$\begin{aligned} \lim_{\epsilon \to 0^+} I_2^\epsilon &= 0, \\ \lim_{\epsilon \to 0^+} I_1^\epsilon &= \lim_{\epsilon \to 0^+} I_{11}^\epsilon + \underbrace{\lim_{\epsilon \to 0^+} I_{12}^\epsilon}_{=0} \\ &= \lim_{\epsilon \to 0^+} \frac{1}{\mathrm{vol}(\partial B_\epsilon(x))} \int_{\partial B_\epsilon(x)} f(y) \mathrm{d}S(y). \end{aligned}$$

The result (3) now follows from the lemma stated right below, whose proof will be a HW. $\square$

**Lemma 10.5.** *For any continuous function $h$:*

$$\lim_{\epsilon\to 0^+} \frac{1}{\mathrm{vol}(\partial B_\epsilon(x))}\int_{\partial B_\epsilon(x)} h(y)\mathrm{d}S(y) = h(x)$$
$$\lim_{\epsilon\to 0^+} \frac{1}{\mathrm{vol}(B_\epsilon(x))}\int_{B_\epsilon(x)} h(y)\mathrm{d}y = h(x)$$

*Proof.* HW □

**Remark 10.6.** From the expression for $\Gamma(x)$ we obtain the following useful estimates:

$$|D\Gamma(x)| \le \frac{C}{|x|^{n-1}}, \quad |D^2\Gamma(x)| \le \frac{C}{|x|^n}, \quad x \neq 0.$$

10.1. **Harmonic functions.**

**Definition 10.7.** A solution to Laplace's equation is called a harmonic function. We say that $u$ is a harmonic function (or simply harmonic) in $\Omega$ if we want to emphasize that it solves Laplace's equation in $\Omega$.

**Theorem 10.8** (mean value formula for Laplace's equation)**.** *Let $u \in C^2(\Omega)$ be harmonic in $\Omega$. Then*

$$u(x) = \frac{1}{\mathrm{vol}(\partial B_r(x))}\int_{\partial B_r(x)} u\mathrm{d}S = \frac{1}{\mathrm{vol}(B_r(x))}\int_{B_r(x)} u\mathrm{d}y,$$

*for each $\overline{B_r(x)} \subset \Omega$.*

**Remark 10.9.** This theorem says that harmonic functions are "non-local" since their value at $x$ depends on their values on $\partial B_r(x)$; in particular $r$ can be arbitrarily large for $\Omega = \mathbb{R}^n$.

*Proof.* Define

$$f(r) := \frac{1}{\mathrm{vol}(\partial B_r(x))}\int_{\partial B_r(x)} u(y)\mathrm{d}S(y).$$

Changing variables $z = \frac{y-x}{r}$, recalling that $\mathrm{d}S = r^{n-1}\mathrm{d}\omega, \mathrm{vol}(\partial B_r(x)) = n\omega_n r^{n-1}$:

$$f(r) = \frac{1}{n\omega_n}\int_{\partial B_1(0)} u(x+rz)\mathrm{d}S(z).$$

Taking the derivative and noticing that we can differentiate under the integral:

$$f'(r) = \frac{1}{n\omega_n}\int_{\partial B_1(0)} \nabla u(x+rz)\cdot z\mathrm{d}S(z).$$

Changing variables back to $y$:

$$f'(r) = \frac{1}{n\omega_n r^{n-1}}\int_{\partial B_r(x)} \nabla u(y)\cdot\left(\frac{y-x}{r}\right)\mathrm{d}S(y).$$

Since $\frac{y-x}{r} = \nu$ =unit outer normal to $\partial B_r(x)$:

$$f'(r) = \frac{1}{n\omega_n r^{n-1}}\int_{\partial B_r(x)} \nabla u(y)\cdot\nu\mathrm{d}S(y)$$
$$= \frac{1}{n\omega_n r^{n-1}}\int_{\partial B_r(x)} \frac{\partial u}{\partial \nu}(y)\mathrm{d}S(y)$$

$$= \frac{1}{n\omega_n r^{n-1}} \int_{B_r(x)} \triangle u(y) \mathrm{d}y = 0$$

where we used Green's identities. Thus, $f(r)$ is constant so

$$\frac{1}{\mathrm{vol}(\partial B_r(x))} \int_{\partial B_r(x)} u(y) \mathrm{d}S(y) = f(r) = \lim_{r \to 0^+} f(r)$$
$$= \lim_{r \to 0^+} \frac{1}{\mathrm{vol}(\partial B_r(x))} \int_{\partial B_r(x)} u(y) \mathrm{d}S(y) = u(x),$$

showing the first equality. For the second, integrate in polar coordinates to find

$$\frac{1}{\mathrm{vol}(B_r(x))} \int_{B_r(x)} u(y) \mathrm{d}y = \frac{1}{\omega_n r^n} \int_0^r \underbrace{\left( \int_{\partial B_s(x)} u \mathrm{d}S \right)}_{= n\omega_n s^{n-1} u(x)} \mathrm{d}s = u(x).$$

□

**Theorem 10.10** (converse of the mean value property)**.** *If $u \in C^2(\Omega)$ is such that*

$$u(x) = \frac{1}{\mathrm{vol}(\partial B_r(x))} \int_{\partial B_r(x)} u \mathrm{d}S$$

*for each $\overline{B_r(x)} \subset \Omega$. Then $u$ is harmonic.*

*Proof.* This will be a HW. □

**Definition 10.11.** Let $U \subset \mathbb{R}^n$. We say that a subset $V \subset U$ is relatively open, or open in $U$, if $V = U \cap W$ for some open set $W \subset \mathbb{R}^n$. $V \subset U$ is said to relatively closed, or closed in $U$, if $V = U \cap W$ for some closed set $W \subset \mathbb{R}^n$. A set $\Omega \subset \mathbb{R}^n$ is called connected if the only non-empty subset of $\Omega$ that is both open and closed in $\Omega$ is $\Omega$ itself.

**Remark 10.12.** Sometimes we say simply that $V \subset U$ is open/closed to mean that is open/closed in $U$, i.e., $U$ is implicitly understood.

Students who have no seen the definition of connected sets are encouraged to think about how the above definition corresponds to the intuition that $\Omega$ cannot be "split into separate pieces. "

**Theorem 10.13** (maximum principle)**.** *Suppose that $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is harmonic where $\Omega$ is bounded. Then*

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u.$$

*Moreover, if $u(x_0) = \max_{\bar{\Omega}} u$ for some $x_0 \in \Omega$, then $u$ is constant.*

**Remark 10.14.** Replacing $u$ by $-u$ we obtain similar statements with min. Thus, we can summarize the maximum principle by saying that a harmonic function achieves its maximum and minimum on the boundary.

*Proof.* Suppose that for some $x_0 \in \Omega$, we have $u(x_0) = M = \max_{\bar{\Omega}} u$. For $0 < r < \mathrm{dist}(x_0, \partial\Omega)$, the mean value property gives:

$$M = u(x_0) = \frac{1}{\mathrm{vol}(B_r(x_0))} \int_{B_r(x_0)} u \mathrm{d}y \leq M.$$

Equality in $\leq$ happens only if $u(y) = M$ for all $y \in B_r(x_0)$. Therefore the set $A := \{x \in \Omega \mid u(x) = M\}$ is both open and closed in $\Omega$, thus, $A = \Omega$, showing the second statement. The first statement follows form the second. $\square$

10.2. **Further results for harmonic functions and Poisson's equation.** Here we list a few important results concerning $\triangle u = f$ that we will not prove.

**Theorem 10.15** (Liouville's Theorem)**.** *Suppose that $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded (i.e., there exists a constant $M \geq 0$ such that $|u(x)| \leq M$ for all $x \in \mathbb{R}^n$). Then $u$ is constant.*

**Definition 10.16.** Let $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ be given. The following boundary-value problem

$$\begin{cases} \triangle u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

is called the (inhomogeneous) Dirichlet problem for the Laplacian.

**Theorem 10.17.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a $C^3$ boundary. Let $f \in C^1(\bar{\Omega})$ and $g \in C^3(\bar{\Omega})$. Then, there exists a unique solution $u \in C^2(\bar{\Omega})$ to the Dirichlet problem*

$$\begin{cases} \triangle u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

**Remark 10.18.** To solve Poisson's equation in $\mathbb{R}^n$ we introduced the fundamental solution. One approach to solve the Dirichlet problem is to introduce an analogue of the fundamental solution which takes the boundary into account, known as the Green function.

## 11. The wave equation in $\mathbb{R}^n$

Here we will study the Cauchy problem for the wave equation in $\mathbb{R}^n$, i.e.,

$$\begin{cases} \Box u = 0 & \text{in } [0,\infty)\times\mathbb{R}^n \\ u = u_0 & \text{on } \{t=0\}\times\mathbb{R}^n \\ \partial_t u = u_1 & \text{on } \{t=0\}\times\mathbb{R}^n \end{cases}$$

where $\Box := -\partial_t^2 + \triangle$ is called the D'Alembertian (or the wave operator) and $u_0, u_1 : \mathbb{R}^n \to \mathbb{R}$ are given. The initial conditions can also be stated as $u(0,x) = u_0(x), \partial_t u(0,x) = u_1(x), x \in \mathbb{R}^n$.

**Definition 11.1.** The sets

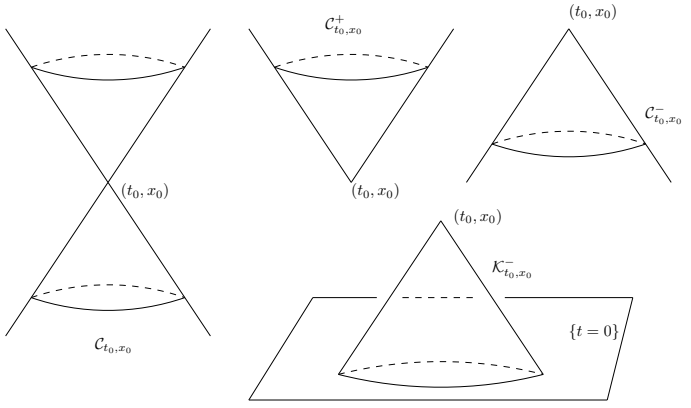
$$\begin{aligned} \mathcal{C}_{t_0,x_0} &:= \{(t,x) \in (-\infty,\infty)\times\mathbb{R}^n \mid |x-x_0| \le |t-t_0|\}, \\ \mathcal{C}^+_{t_0,x_0} &:= \{(t,x) \in (-\infty,\infty)\times\mathbb{R}^n \mid |x-x_0| \le t-t_0\}, \\ \mathcal{C}^-_{t_0,x_0} &:= \{(t,x) \in (-\infty,\infty)\times\mathbb{R}^n \mid |x-x_0| \le t_0-t\}, \end{aligned}$$

are called, respectively, the light-cone, future light-cone, and past light-cone with vertex at $(t_0,x_0)$. The sets

$$\begin{aligned} \mathcal{K}_{t_0,x_0} &:= \mathcal{C}_{t_0,x_0} \cap \{t \ge 0\} \\ \mathcal{K}^+_{t_0,x_0} &:= \mathcal{C}^+_{t_0,x_0} \cap \{t \ge 0\}, \\ \mathcal{K}^-_{t_0,x_0} &:= \mathcal{C}^-_{t_0,x_0} \cap \{t \ge 0\}, \end{aligned}$$

are called, respectively, the light-cone, future light-cone, and past light-cone for positive time with vertex at $(t_0,x_0)$.

We often omit "for positive time" and refer to the sets $\mathcal{K}$ as light-cones. We also refer to a part of a cone, e.g., for $0 \le t \le T$, as the truncated (future, past) light-cone.




**Lemma 11.2** (differentiation of moving regions)**.** *Let $\Omega(\tau) \subset \mathbb{R}^n$ be a family of bounded domains with smooth boundary depending smoothly on the parameter $\tau$. Let $v$ be the velocity of the moving boundary $\partial\Omega(\tau)$ and $\nu$ the unit outer normal to $\partial\Omega(\tau)$. If $f = t(\tau,x)$ is smooth then*

$$\frac{d}{d\tau}\int_{\Omega(\tau)} f\,\mathrm{d}x = \int_{\Omega(\tau)} \partial_\tau f\,\mathrm{d}x + \int_{\partial\Omega(\tau)} f v\cdot\nu\,\mathrm{d}S.$$

*Proof.* HW. (Compare this with the fundamental theorem of calculus). □

**Theorem 11.3** (finite propagation speed)**.** *Let $u \in C^2([0,\infty)\times\mathbb{R}^n)$ be a solution to the Cauchy problem for the wave equation. If $u_0 = u_1 = 0$ on $\{t=0\}\times B_{t_0}(x_0)$, then $u = 0$ within $\mathcal{K}^-_{t_0,x_0}$. (Thus, the solution at $(t_0,x_0)$ depends only on the data on $B_{t_0}(x_0)$ and the cone $\mathcal{K}^-_{t_0,x_0}$ is also called a domain of dependence).*

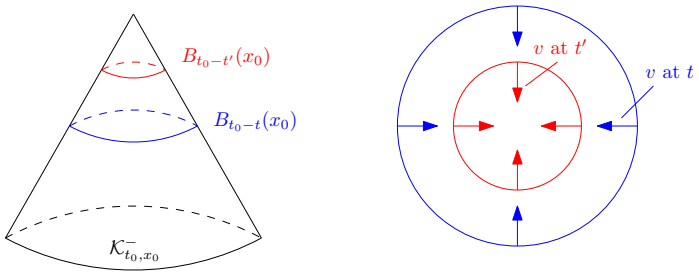
*Proof.* Define the "energy" as

$$E(t) = \frac{1}{2}\int_{B_{t_0-t}(x_0)} ((\partial_t u)^2 + |\nabla u|^2)\mathrm{d}x, \quad 0 \le t \le t_0.$$

Then:

$$\frac{dE}{dt} = \int_{B_{t_0-t}(x_0)} (\partial_t u \partial_t^2 u + \nabla u\cdot\nabla\partial_t u)\mathrm{d}x + \frac{1}{2}\int_{\partial B_{t_0-t}(x_0)} ((\partial_t u)^2 + |\nabla u|^2) v\cdot\nu \mathrm{d}S.$$

The points on the boundary move inward orthogonally to the spheres $\partial B_{t_0-t}(x_0)$ and with finite speed linear in $t$, thus $v = -\nu$.




Integrating by parts:

$$\int_{B_{t_0-t}(x_0)} \nabla u\cdot\nabla\partial_t u\mathrm{d}x = -\int_{B_{t_0-t}(x_0)} \triangle u\partial_t u\mathrm{d}x + \int_{\partial B_{t_0-t}(x_0)} \frac{\partial u}{\partial\nu}\partial_t u\mathrm{d}S.$$

Thus,

$$\begin{aligned}\frac{dE}{dt} &= \int_{B_{t_0-t}(x_0)} \underbrace{(\partial_t^2 u - \triangle u)}_{=0}\partial_t u\mathrm{d}S + \int_{\partial B_{t_0-t}(x_0)} \frac{\partial u}{\partial\nu}\partial_t u - \frac{1}{2}\int_{\partial B_{t_0-t}(x_0)} ((\partial_t u)^2 + |\nabla u|^2)\mathrm{d}S \\ &= \int_{\partial B_{t_0-t}(x_0)} \left(\frac{\partial u}{\partial\nu}\partial_t u - \frac{1}{2}(\partial_t u)^2 - \frac{1}{2}|\nabla u|^2\right)\mathrm{d}S \\ &\le \int_{\partial B_{t_0-t}(x_0)} \left(|\nabla u||\partial_t u| - \frac{1}{2}(\partial_t u)^2 - \frac{1}{2}|\nabla u|^2\right)\mathrm{d}S,\end{aligned}$$

where we used that $\frac{\partial u}{\partial\nu}\partial_t u \le \left|\frac{\partial u}{\partial\nu}\partial_t u\right| = \left|\frac{\partial u}{\partial\nu}\right||\partial_t u|$ and

$$\left|\frac{\partial u}{\partial\nu}\right| = |\nabla\cdot\nu| \le |\nabla u|\underbrace{|\nu|}_{=1} = |\nabla u|.$$

Now apply the Cauchy-Schwarz inequality $ab \le \frac{a^2}{2} + \frac{b^2}{2}$ with $a = |\nabla u|, b = |\partial_t u|$, to get

$$\frac{dE}{dt} \le \int_{\partial B_{t_0-t}(x_0)} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}(\partial_t u)^2 - \frac{1}{2}(\partial_t u)^2 - \frac{1}{2}|\nabla u|^2\right) = 0,$$

thus $E(t)$ is decreasing. Since $E(t) \geq 0$ and

$$E(0) = \frac{1}{2}\int_{\partial B_{t_0}(x_0)} (\underbrace{(\partial_t u(0,x))}_{=u_1(x)=0})^2 + \underbrace{|\nabla u(0,x)|^2}_{|\nabla u_0(0,x)|=0})\mathrm{d}x = 0.$$

We conclude that $E(t) = 0$ for all $0 \leq t \leq t_0$.

Since $E(t)$ is the integral of a positive continuous function over $B_{t-t_0}(x_0)$, $E(t) = 0$ implies that, for each $t$, the integrand must vanish, i.e.,

$$(\partial_t u(t,x))^2 + |\nabla u(t,x)|^2 = 0$$

for all $(t,x) \in \mathcal{K}^-_{t_0,x_0}$, which then implies $\partial_t u(t,x) = 0$ and $\nabla u(t,x) = 0$ for all $(t,x) \in \mathcal{K}^-_{t_0,x_0}$. Since $\mathcal{K}^-_{t_0,x_0}$ is connected, we conclude that $u$ is constant in time and space within $\mathcal{K}^-_{t_0,x_0}$, i.e., $u(t,x) = C$ =constant in $\mathcal{K}^-_{t_0,x_0}$. Since $u(0,x) = u_0(x) = 0$, $C$ must be zero. □

**Notation 11.4.** Henceforth, we assume that $n \geq 2$. Set

$$U(t,x;r) := \frac{1}{\mathrm{vol}(\partial B_r(x))}\int_{\partial B_r(x)} u(t,y)\mathrm{d}S(y),$$
$$U_0(x;r) := \frac{1}{\mathrm{vol}(\partial B_r(x))}\int_{\partial B_r(x)} u_0(t,y)\mathrm{d}S(y),$$
$$U_1(x;r) := \frac{1}{\mathrm{vol}(\partial B_r(x))}\int_{\partial B_r(x)} u_1(t,y)\mathrm{d}S(y)$$

which are spherical average over $\partial B_r(x)$.

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