HOMEWORK 6 SOLUTIONS

MATH 3120

Unless stated otherwise, the notation below is as in class. You can assume that all functions are C^∞ unless stated otherwise.

Question 1. Use Duhamel's principle to show that a solution to the inhomoegeneous wave equation in $1d$ with zero data and source term f is give by

$$u(t,x) = \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(t-s,y)\,dyds. \tag{1}$$

To do so, first use D'Alembert's formula to conclude that

$$u_s(t,x) = \frac{1}{2}\int_{x-t+s}^{x+t-s} f(s,y)\,dy.$$

Use the definition of u in terms of u_s and change variables to conclude (1).

Solution 1. Using D'Alembert's formula, we find

$$u_s(t,x) = \frac{1}{2}\int_{x-t+s}^{x+t-s} f(s,y)\,dy,$$

where we used the fact that D'Alembert's formula was derived for data at $t = 0$; for data at $t = s$ we have to replace t by $t - s$ in the limits of integration. Thus

$$u(t,x) = \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(s,y)\,dyds = \frac{1}{2}\int_0^t \int_{x-z}^{x+z} f(t-z,y)\,dydz,$$

where we made the change $s = t - z$.

Question 2. Use Duhamel's principle to show that a solution to the inhomoegeneous wave equation in $3d$ with zero data and source term f is give by

$$u(t,x) = \frac{1}{4\pi}\int_{B_t(x)} \frac{f(t-|y-x|,y)}{|y-x|}\,dy. \tag{2}$$

(The integrand in (2) is known as the retarded potential.) To do so, first use Kirchhoff's formula for solutions in $n = 3$ to conclude that

$$u_s(t,x) = \frac{t-s}{\operatorname{vol}(\partial B_{t-s}(x))}\int_{\partial B_{t-s}(x)} f(s,y)\,dS(y).$$

Use the definition of u in terms of u_s and change variables to conclude (2).

Solution 2. Kirchhoff's formula gives

$$u_s(t,x) = \frac{1}{\operatorname{vol}(\partial B_{t-s}(x))}\int_{\partial B_{t-s}(x)} (t-s)f(s,y)\,dS(y).$$

Thus

$$\begin{aligned}
u(t,x) &= \int_0^t \frac{t-s}{\operatorname{vol}(\partial B_{t-s}(x))} \int_{\partial B_{t-s}(x)} f(s,y)\, dS(y)ds \\
&= \frac{1}{4\pi} \int_0^t \int_{\partial B_{t-s}(x)} \frac{f(s,y)}{t-s}\, dS(y)ds \\
&= \frac{1}{4\pi} \int_0^t \int_{\partial B_r(x)} \frac{f(t-r,y)}{r}\, dS(y)dr \\
&= \frac{1}{4\pi} \int_{B_t(x)} \frac{f(t-|y-x|,y)}{|y-x|}\, dy,
\end{aligned}$$

where we made the change of variables $r = t - s$ and then wrote $r = |y - x|$.

Question 3. Show that there exists a constant $C > 0$ such that for any solution u to the $3d$ wave equation it holds that

$$|u(t,x)| \leq \frac{C}{t} \int_{\mathbb{R}^3} (|D^2 u_0(y)| + |Du_0(y)| + |u_0(y)| + |Du_1(y)| + |u_1(y)|)\, dy$$

for $t \geq 1$.

Hint: Use Kirchhoff's formula, note that for any function f we have

$$f(y) = f(y) \frac{y-x}{t} \cdot \frac{y-x}{t}$$

on $\partial B_t(x)$, and use one of Green's identities.

Solution 3. We have

$$u(t,x) = \frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} (u_0(y) + t u_1(y) + \nabla u_0(y) \cdot (y-x))\ dS(y).$$

The unit outer normal to $\partial B_t(x)$ is $\nu = (y-x)/t$, so that $\nu \cdot \nu = \frac{y-x}{t} \cdot \frac{y-x}{t} = 1$. Therefore, using this and Green's identities,

$$\begin{aligned}
\frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} u_0(y)\, dS(y) &= \frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} u_0(y) \nu \cdot \frac{y-x}{t}\, dS(y) \\
&= \frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{B_t(x)} \operatorname{div}_y \left(u_0(y) \frac{y-x}{t} \right)\ dy \\
&= \frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{B_t(x)} \left(\nabla u_0(y) \cdot \frac{y-x}{t} + u_0(y) \frac{3}{t} \right)\ dy,
\end{aligned}$$

so that

$$\begin{aligned}
\left| \frac{1}{\operatorname{vol}(\partial B_t(x))} \int_{\partial B_t(x)} u_0(y)\, dS(y) \right| &\leq \frac{C}{t^2} \int_{B_t(x)} (|\nabla u_0(y)| + |u_0(y)|)\ dy \\
&\leq \frac{C}{t^2} \int_{\mathbb{R}^3} (|\nabla u_0(y)| + |u_0(y)|)\ dy.
\end{aligned}$$

A similar inequality holds for the u_1 integral (with an extra factor of t), and for ∇u_0:

$$\begin{aligned}\frac{1}{\mathrm{vol}(\partial B_t(x))}\int_{\partial B_t(x)} \nabla u_0(y)\cdot(y-x)\,dS(y) &= \frac{t}{\mathrm{vol}(\partial B_t(x))}\int_{\partial B_t(x)} \nabla u_0(y)\cdot\nu\,dS(y)\\ &= \frac{1}{4\pi t}\int_{B_t(x)} \Delta u_0(y)\,dy,\end{aligned}$$

so that

$$\left|\frac{1}{\mathrm{vol}(\partial B_t(x))}\int_{\partial B_t(x)} \nabla u_0(y)\cdot(y-x)\,dS(y)\right| \le \frac{C}{t}\int_{\mathbb{R}^3} |D^2 u_0(y)|\,dy.$$

Combining the foregoing produces the result.