

## HOMEWORK 5 SOLUTIONS

MATH 3120

Unless stated otherwise, the notation below is as in class. You can assume that all functions are  $C^\infty$  unless stated otherwise.

**Question 1.** Prove the differentiation of moving regions formula stated in class:

$$\frac{d}{d\tau} \int_{\Omega(\tau)} f \, dx = \int_{\Omega(\tau)} \partial_\tau f \, dx + \int_{\partial\Omega(\tau)} f v \cdot \nu \, dS. \quad (1)$$

(See the class notes for the notation and precise assumptions.) For simplicity, prove (1) in the following particular case. Assume that  $n = 3$  and that the domains  $\Omega(\tau)$  are given by a one-parameter family of one-to-one and onto maps  $\varphi = \varphi(\tau, x) : \Omega \rightarrow \Omega(\tau) = \varphi(\tau, \Omega)$ , where  $\Omega := \Omega(0)$  and  $\varphi(0, \cdot) = \text{id}_\Omega$ , where  $\text{id}_\Omega$  is the identity map on  $\Omega$ , i.e.,  $\text{id}_\Omega(x) = x$ ,  $x \in \Omega$ .

(a) For each fixed  $\tau$ , consider the change of variables  $x = \varphi(\tau, y)$ , so that

$$\int_{\Omega(\tau)} f(\tau, x) \, dx = \int_{\Omega} f(\tau, \varphi(\tau, y)) J(\tau, y) \, dy, \quad (2)$$

where  $J(\tau, y)$  is the Jacobian of the transformation  $x = \varphi(\tau, y)$  for fixed  $\tau$ .

(b) Show that there exists a one parameter family of vector fields  $u(\tau, \cdot)$  such that

$$\partial_\tau \varphi(\tau, x) = u(\tau, \varphi(\tau, x)).$$

(c) Explain why  $u = v$  on  $\partial\Omega(\tau)$ .

(d) Show that

$$\partial_\tau J(\tau, x) = (\text{div } u)(\tau, \varphi(\tau, x)) J(\tau, x).$$

(e) Use (2) and the above to compute  $\frac{d}{d\tau} \int_{\Omega(\tau)} f$ , and do an integration by parts to obtain the result.

**Solution 1.** (a) This is simply the change of variables formula from calculus.

(b) For each fixed  $x$ , the map  $\tau \mapsto \varphi(\tau, x)$  is a curve in  $\mathbb{R}^3$ .  $\partial_\tau \varphi(\tau, x)$  is, therefore, the tangent vector to this curve at  $\varphi(\tau, x)$  at time  $\tau$ . The collection of all such tangent vectors, as  $\tau$  and  $x$  vary, forms the vector field  $u$ .

(c) The map  $\varphi$  sends  $\partial\Omega$  onto  $\partial\Omega(\tau)$  for each  $\tau$ . Since  $\partial_\tau \varphi(\tau, x)$  is the velocity at time  $\tau$  of the particle that started at  $x \in \Omega$  at time zero,  $u(\tau, \varphi(\tau, x))$  is the velocity of  $\partial\Omega(\tau)$  at the point  $\varphi(\tau, x) \in \partial\Omega(\tau)$ .

(d) According to the notation of part (a), we set

$$\varphi_j^i = \frac{\partial}{\partial y^j} \varphi^i, \quad \partial_j u^i = \frac{\partial}{\partial x^j} u^i,$$

where we considered  $\varphi = (\varphi^1, \varphi^2, \varphi^3)$ . In particular, note that when we write  $\varphi_j^i = \partial_j \varphi^i$  the derivative is always with respect to  $y \in \Omega$ , whereas when we write  $\partial_j u^i$  the derivative is always with respect to  $x \in \Omega(\tau)$ .

Recall the following formula for the determinant of a  $n \times n$  matrix  $a$  with entries  $a_j^i = a_{\text{column}}^{\text{row}}$ :

$$\det(a) = \frac{1}{n!} \epsilon_{i_1 \dots i_n} \epsilon^{j_1 \dots j_n} a_{j_1}^{i_1} \dots a_{j_n}^{i_n}.$$

In our case, this gives

$$J(\tau, y) = \frac{1}{3!} \epsilon_{i_1 i_2 i_3} \epsilon^{j_1 j_2 j_3} \varphi_{j_1}^{i_1} \varphi_{j_2}^{i_2} \varphi_{j_3}^{i_3}.$$

Recall that the definition of  $J$  involves an absolute value, which we can omit here since  $J > 0$  because  $J(0, \cdot) > 0$ . Compute

$$\begin{aligned} \partial_\tau \varphi_j^i &= \partial_j \partial_\tau \varphi^i \\ &= \frac{\partial}{\partial y^j} u^i \\ &= \partial_\ell u^i \varphi_j^\ell, \end{aligned}$$

where in the second equality we used (b) and in the third one the chain rule. Therefore

$$J(\tau, y) = \frac{1}{3!} \epsilon_{i_1 i_2 i_3} \epsilon^{j_1 j_2 j_3} (\partial_\ell u^{i_1} \varphi_{j_1}^\ell \varphi_{j_2}^{i_2} \varphi_{j_3}^{i_3} + \varphi_{j_1}^{i_1} \partial_\ell u^{i_2} \varphi_{j_2}^\ell \varphi_{j_3}^{i_3} + \varphi_{j_1}^{i_1} \varphi_{j_2}^{i_2} \partial_\ell u^{i_3} \varphi_{j_3}^\ell). \quad (3)$$

Because  $\epsilon_{i_1 i_2 i_3}$  is non-zero only for  $i_1 i_2 i_3$  all different from each other, for each triple  $i_1 i_2 i_3$ , the term  $\epsilon_{i_1 i_2 i_3} \partial_\ell u^{i_1} \varphi_{j_1}^\ell \varphi_{j_2}^{i_2} \varphi_{j_3}^{i_3}$  is non-zero only when  $\ell = i_1$ . Similarly for the second and third terms on the RHS of (3), and we obtain

$$J(\tau, y) = \frac{1}{3!} \sum_{\substack{i_1, i_2, i_3=1 \\ j_1, j_2, j_3=1}}^3 \epsilon_{i_1 i_2 i_3} \epsilon^{j_1 j_2 j_3} (\partial_{i_1} u^{i_1} + \partial_{i_2} u^{i_2} + \partial_{i_3} u^{i_3}) \varphi_{j_1}^{i_1} \varphi_{j_2}^{i_2} \varphi_{j_3}^{i_3}.$$

Because the summand is non-zero only if  $i_1 i_2 i_3$  are all different from each other, the term in parenthesis is always equal to  $\partial_1 u^1 + \partial_2 u^2 + \partial_3 u^3 = \operatorname{div} u$ , which gives the result.

(e) We have

$$\begin{aligned}
\frac{d}{d\tau} \int_{\Omega(\tau)} f \, dx &= \partial_\tau \int_{\Omega} f(\tau, \varphi(\tau, y)) J(\tau, y) \, dy \\
&= \int_{\Omega} (\partial_\tau f(\tau, \varphi(\tau, y)) J(\tau, y) + \nabla f(\tau, \varphi(\tau, y)) \cdot \partial_\tau \varphi(\tau, y) J(\tau, y) + f(\tau, \varphi(\tau, y)) \partial_\tau J(\tau, y)) \, dy \\
&= \int_{\Omega} (\partial_\tau f(\tau, \varphi(\tau, y)) J(\tau, y) + \nabla f(\tau, \varphi(\tau, y)) \cdot u(\tau, \varphi(\tau, y)) J(\tau, y) \\
&\quad + f(\tau, \varphi(\tau, y)) (\operatorname{div} u)(\tau, \varphi(\tau, y)) J(\tau, y)) \, dy \\
&= \int_{\Omega(\tau)} (\partial_\tau f(\tau, x) + \nabla f(\tau, x) \cdot u(\tau, x) + f(\tau, x) (\operatorname{div} u)(\tau, x)) \, dx \\
&= \int_{\Omega(\tau)} (\partial_\tau f(\tau, x) - f(\tau, x) (\operatorname{div} u)(\tau, x) + f(\tau, x) (\operatorname{div} u)(\tau, x)) \, dx \\
&\quad + \int_{\partial\Omega(\tau)} f(\tau, x) u(\tau, x) \cdot \nu(\tau, x) \, dS(x) \\
&= \int_{\Omega(\tau)} \partial_\tau f(\tau, x) \, dx + \int_{\partial\Omega(\tau)} f(\tau, x) v(\tau, x) \cdot \nu(\tau, x) \, dS(x).
\end{aligned}$$

Above, we the steps are as follows: in the second line we used the product rule and the chain rule; in the third line we used (b) and (d); on the fourth line, we changed variables back to  $x$ ; on the fifth line we integrated  $\nabla f$  by parts (equivalently, used one of the Green identities); on the last line, we used (c).

**Question 2.** Let  $u$  be a solution to the Cauchy problem for the wave equation in  $\mathbb{R}^n$ . Assume that  $u_0$  and  $u_1$  have their supports in the ball  $B_R(0)$  for some  $R > 0$ . Show that  $u = 0$  in the exterior of the region

$$I := \{(t, x) \in (0, \infty) \times \mathbb{R}^n \mid x \in B_{R+t}(0)\}.$$

$I$  is called a domain of influence for that data on  $B_R(0)$  (compare with the 1d case).

**Solution 2.** Let  $(t, x) \notin I$ . Then  $K_{t,x}^- \cap I = \emptyset$ , and the result follows from the finite-propagation speed for the wave equation.

**Question 3.** Let  $u$  be a solution to the Cauchy problem for the wave equation and assume that  $u_0$  and  $u_1$  have compact support.

(a) Show that the energy

$$E(t) := \frac{1}{2} \int_{\mathbb{R}^n} [(\partial_t u)^2 + |\nabla u|^2] \, dx$$

is well-defined.

(b) Show that

$$E(t) = E(0),$$

i.e., the energy is conserved.

**Question 4.** (a) By question 2, the solution  $u$  has compact support for each fixed  $t$ .

(b) For each  $t_0$  and  $\varepsilon > 0$ , there exists, by (a), a  $R_* > 0$  such that  $u(t, x) = 0$  for all  $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$  and  $|x| \geq R_*$ . We now follow the proof of the finite-propagation speed property for the wave equation (see the class notes) using the ball  $B_{R_*}$ , and observe the following. In that proof, we did an integration by parts, and controlled the boundary term using the Cauchy-Schwarz inequality. Here, this boundary term vanishes identically by the foregoing. We obtain therefore a sequence of equalities (rather than inequalities as in the proof done in class), which then gives the result.

**Question 5.** Let  $u$  be a solution to the Cauchy problem for the wave equation in  $\mathbb{R}^3$  with compactly supported data (i.e.,  $u_0$  and  $u_1$  have compact support).

(a) Show that there exists a constant  $C > 0$ , depending on  $u_0$  and  $u_1$ , such that

$$|u(t, x)| \leq \frac{C}{t}, \quad (4)$$

for  $t \geq 1$  and  $x \in \mathbb{R}^3$ . Thus, for each fixed  $x$ ,  $u$  approaches zero as  $t \rightarrow \infty$ , i.e., solutions decay in time.

*Hint:* Use the formula for solutions in  $n = 3$ . Since the data has compact support, it vanishes outside  $B_R(0)$  for some  $R > 0$ . This implies an estimate for the area of the largest region within  $B_t(x)$  where the data is non-trivial.

(b) Is the estimate (4) sharp? (I.e., can it be improved to show that solutions decay faster in time than  $\frac{1}{t}$ ?)

(c) Do we still get decay if the data does not have compact support?

**Solution 3.** (a) The solution is given by

$$u(t, x) = \frac{1}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x)} (u_0(y) + tu_1(y) + \nabla u_0(y) \cdot (y - x)) dS(y).$$

Since the data is compactly supported, there exists a  $R > 0$  such that  $u_0(x) = 0$  and  $u_1(x) = 0$  for  $|x| \geq R$ , so that

$$u(t, x) = \frac{1}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_R(0)} (u_0(y) + tu_1(y) + \nabla u_0(y) \cdot (y - x)) dS(y).$$

Because the data is compactly supported, we have  $|u_0|, |u_1|, |\nabla u_0| \leq C$  for some  $C > 0$ , so that

$$\begin{aligned} |u(t, x)| &\leq \frac{C}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_R(0)} (1 + t + |y - x|) dS \\ &= \frac{C}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_R(0)} \left(1 + t + \frac{t|y - x|}{t}\right) dS \\ &\leq \frac{C}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_R(0)} (1 + t + t) dS \\ &\leq \frac{C(1 + t)}{t^2} \int_{\partial B_t(x) \cap B_R(0)} dS, \end{aligned}$$

where we used that  $|y - x|/t = 1$  since  $y \in B_t(x)$  and that  $\text{vol}(\partial B_t(x)) = 4\pi t^2$ . Because  $\partial B_t(x) \cap B_R(0)$  has area at most  $4\pi R^2$ , we have the result.

(b) Yes, it cannot be improved for arbitrary solutions of the wave equation. To see this, take  $u_0 = 0$  and  $u_1$  to be a non-negative compactly supported function that is equal to 1 on  $B_1(0)$ . Then

$$\begin{aligned} u(t, x) &= \frac{1}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x)} t u_1(y) dS(y) \\ &= \frac{t}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_1(0)} u_1(y) dS(y) + \frac{t}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x) \setminus (B_t(x) \cap B_1(0))} u_1(y) dS(y). \end{aligned}$$

Note that the second term on the RHS is always non-negative, thus

$$u(t, x) \geq \frac{t}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_1(0)} u_1(y) dS(y) = \frac{t}{\text{vol}(\partial B_t(x))} \int_{\partial B_t(x) \cap B_1(0)} dS.$$

For any  $x$  on the boundary of the lightcone, i.e.,  $|x| = t$ , and such that  $|x| \geq 1$ , we have that the area of  $\partial B_t(x) \cap B_1(0)$  is  $\geq C > 0$ , so that  $u(t, x) \geq C/t$ .

(c) Not necessarily, e.g., take  $u_0 = 0$  and  $u_1 = 1$ , then  $u(t, x) = t$  is the solution.