HOMEWORK 4 SOLUTIONS

MATH 3120

Unless stated otherwise, the notation below is as in class.

Question 1. Prove the following fact that we used in the construction of solutions to Poisson's equation: let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous, then

$$\lim_{r\to 0^+} \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} f\, dS = f(x).$$

Hint: Consider the difference $f(x) - \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} f\, dS$ and use $\frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} dS = 1$.

Solution 1. We have to prove that given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < r < \delta$ then

$$\left| \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} f\, dS - f(x) \right| < \varepsilon.$$

Write

$$\begin{aligned}
\frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} f(y)\, dS(y) - f(x) &= \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} f(y)\, dS - \frac{f(x)}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} dS(y) \\
&= \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} (f(y) - f(x))\, dS(y).
\end{aligned}$$

Thus

$$\left| \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} f(y)\, dS(y) - f(x) \right| \leq \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} |f(y) - f(x)|\, dS(y).$$

Fix $\varepsilon > 0$. Since f is continuous, there exists a $\delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. If $r < \delta$, then $|x - y| < \delta$ for all $y \in \partial B_r(x)$, thus

$$\left| \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} f(y)\, dS(y) - f(x) \right| < \frac{1}{\operatorname{vol}(\partial B_r(x))} \int_{\partial B_r(x)} \varepsilon\, dS = \varepsilon.$$

Question 2. In class, we constructed solutions to Poisson's equation in $\mathbb{R}^n$ for $n \geq 3$. Carry out the construction in the case $n = 2$. You do *not* have to do all the steps. Rather, follow what was done in class and point out what changes in $n = 2$. This boils down to slightly modifying some of the estimates for the fundamental solution.

Solution 2. We use the following estimates in the $n = 2$ case:

$$\int_{B_\varepsilon(0)} |\Gamma(y)|\, dy \leq C\varepsilon^2 |\ln \varepsilon| \to 0 \text{ as } \varepsilon \to 0^+,$$

and

$$\int_{\partial B_\varepsilon(0)} |\Gamma(y)|\, dS(y) \leq C\varepsilon |\ln \varepsilon| \to 0 \text{ as } \varepsilon \to 0^+,$$

and the rest of the proof is essentially the same.

Question 3. Let u be a non-trivial harmonic function in $\mathbb{R}^n$. Can u have compact support? *Hint:* mean value theorem.

Solution 3. No. Let u be harmonic and with compact support and fix an arbitrary $x \in \mathbb{R}^n$. By the compact support property, there exists a $r > 0$ such that $u(y) = 0$ for all $y \in \partial B_r(x)$. By the mean value formula

$$u(x) = \frac{1}{\text{vol}(\partial B_r(x))} \int_{\partial B_r(x)} u(y)\, dS(y) = 0,$$

so that $u = 0$ since x is arbitrary.

Question 4. Prove the converse of the mean value theorem. I.e., let $u \in C^2(\Omega)$ be such that

$$u(x) = \frac{1}{\text{vol}(\partial B_r(x))} \int_{\partial B_r(x)} u\, dS$$

for each $B_r(x) \subset \Omega$. Show that $\Delta u = 0$ in Ω.
Hint: Assume that $\Delta u(x) \neq 0$ for some $x \in \Omega$. Use the functions $f(r)$, $f'(r)$ used in the proof of the mean value to derive a contradiction.

Solution 4. If u is not harmonic, there exists a $x \in \Omega$ such that $\Delta u(x) \neq 0$. By assumption, the function

$$f(r) = \frac{1}{\text{vol}(\partial B_r(x))} \int_{\partial B_r(x)} u\, dS$$

is constant equal to $u(x)$ on the interval $(0, R)$, where $R > 0$ is a fixed number such that $B_R(x) \subset \Omega$. Thus $f'(r) = 0$ for all $r \in (0, R)$. On the other hand, by continuity, Δu has a definite sign, say positive, on a ball $B_{r_0}(x)$ for some $r_0 > 0$, which without loss of generality we can take such that $r_0 < R$. Arguing as in the proof of the mean value theorem, we find

$$f'(r_0) = \frac{1}{n\omega_n r_0^{n-1}} \int_{B_{r_0}(x)} \Delta u(y)\, dy > 0,$$

contradicting $f'(r_0) = 0$.

The next questions are optional, i.e., they will not be graded. They are intended to help students who may have some difficulties with the formal aspects of PDEs discussed in class.

Question 5. Write each PDE below in the form $F(x, u, Du, \dots, D^m u) = 0$, i.e., identify the function F. State if the PDE is homogeneous or non-homogeneous, linear or non-linear.

(a) $u_{tt} - u_{xx} = f$.

(b) $u_y + u u_x = 0$.

(c) $a^{ijk} \partial^3_{ijk} v + v = 0$,

where i, j, k range from 1 to 3.

(d) $u_{xx} + x^2 y^2 u_{yy} = (x + y)^2$.

(e) $u_{xy} + \cos(u) = \sin(xy)$.

Question 6. Consider a PDE $F(x, u, Du, \ldots, D^m u) = 0$. Prove that the PDE is linear (as defined in class in terms of linearity of F with respect to some of its entries) if and only if it can be written as

$$\sum_{|\alpha| \leq k} a_\alpha D^\alpha u = f,$$

as stated in class.