HOMEWORK 3

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

Question 1. The goal of this problem is to prove the following theorem stated in class: Let $g, h \in C^2([0,L])$ satisfy $g(0) = g(L) = 0 = h(0) = h(L)$ and $g''(0) = g''(L) = 0 = h''(0) = h''(L)$. Then, the formal solution

$$u(t,x) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi c t}{L} + b_n \sin \frac{n\pi c t}{L} \right) \sin \frac{n\pi x}{L},$$

where a_n and b_n are given by

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx, \\ b_n &= \frac{2}{n\pi c} \int_0^L h(x) \sin \frac{n\pi x}{L}\, dx, \end{aligned}$$

is a C^2 solution of the initial-boundary value problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, && \text{in } (0,\infty) \times (0,L), \\ u(t,0) = u(t,L) &= 0, && t \geq 0, \\ u(0,x) &= g(x), && 0 \leq x \leq L, \\ \partial_t u(0,x) &= h(x), && 0 \leq x \leq L, \end{aligned}$$

To prove the theorem, proceed as follows.

(a) Show that g and h can be extended to $2L$-periodic C^2 odd functions on $\mathbb{R}$. Call these extensions $\tilde{g}$ and $\tilde{h}$.

(b) Use D'Alembert's formula to solve the initial value problem for the wave equation on $\mathbb{R}$ with data $\tilde{g}$ and $\tilde{h}$. (In class we derived D'Alembert's formula with $c = 1$; here you need the formula for a general c.)

(c) Consider the Fourier series for $\tilde{g}$ and $\tilde{h}$. Plug these into D'Alembert's formula and using trigonometric identities arrive at the expression given by the formal solution for $x \in [0,L]$. Observe that the boundary conditions are satisfied.

(d) In all the above, make sure that you have the correct assumptions to guarantee the convergence of the Fourier series you employ and whatever other theorem you may need to invoke.

Question 2. In class we saw that if $u_0 \in C^2(\mathbb{R})$ and $u_1 \in C^1(\mathbb{R})$, then the Cauchy problem for the $1d$ wave equation with data (u_0, u_1) admits a unique C^2 solution. What can you say if $u_0 \in C^k(\mathbb{R})$ and $u_0 \in C^{k-1}(\mathbb{R})$, $k > 2$?

Question 3. In class we solved the $1d$ wave equation for $t \geq 0$. Making a change of variables $t \mapsto -t$, show that we can also solve the wave equation for negative times. Conclude that D'Alembert's formula is valid for $-\infty < t < \infty$.

Question 4. Consider the Cauchy problem for the $1d$ wave equation with data (u_0, u_1) and then with data $(\tilde{u}_0, \tilde{u}_1)$. Let u and $\tilde{u}$ be the corresponding solutions. Assume that on $[a, b]$ we have $u_0 = \tilde{u}_0$ and $u_1 = \tilde{u}_1$ Prove that $u = \tilde{u}$ in the domain of dependence with base $[a, b]$.

Question 5. In the first question, drop the hypothesis $g''(0) = g''(L) = 0 = h''(0) = h''(L)$. What can you say about the formal solution in this case (will it be an actual solution in any sense? Classical, generalized?)

The next questions are optional, i.e., they will not be graded. They are intended to help students who may have some difficulties with the section "Some general tools, definitions, and conventions for the study of PDEs." The notation is the same used in that section.

Question 6. In class we said Ω has a C^k boundary if $\partial\Omega$ can be written locally as the graph of a C^k function. Make this definition more precise upon using mathematical quantifiers.

Question 7. Show that if a_{ij} is symmetric in i and j, then $a^i{}_j = a^j{}_i$ so that we can write simply a^i_j, but that this is not the case otherwise.

Question 8. Let a be a $n \times n$ matrix with entries $a^i{}_j$, where i the row and j the column. Show that the trace of a is given by $a^i{}_i$. If a is invertible with entries $(a^{-1})^i{}_j$, show that

$$a^i{}_j (a^{-1})^j{}_\ell = \delta^i_\ell.$$

Show that the determinant of a is given by

$$\det(a) = \frac{1}{n!} \epsilon_{i_1 i_2 \cdots i_n} \epsilon^{j_1 j_2 \cdots j_n} a^{i_1}{}_{j_1} a^{i_2}{}_{j_2} \cdots a^{i_n}{}_{j_n}.$$

Above, $\epsilon_{i_1 i_2 \cdots i_n}$ is the n-dimensional totally anti-symmetric symbol, defined as $\epsilon_{i_1 i_2 \cdots i_n} = 1$ if $i_1, i_2, \cdots, i_n$ is an even permutation of $1, 2, \cdots, n$, $\epsilon_{i_1 i_2 \cdots i_n} = -1$ if $i_1, i_2, \cdots, i_n$ is an odd permutation of $1, 2, \cdots, n$, and $\epsilon_{i_1 i_2 \cdots i_n} = 0$ otherwise.
Hint: Show the determinant formula only for $n = 2$ and, perhaps, $n = 3$. The general case is too lengthy for you to spend time on this. You can, however, see the general proofs in textbooks if you are interested.

Question 9. Assuming the integration by parts formula, prove the Green identities stated in class.