HOMEWORK 2

MATH 3120

Unless stated otherwise, the notation and terminology below is the same used in class.

Question 1. Show that in spherical coordinates the Laplacian reads

$$\Delta = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta_{S^2},$$

where

$$\Delta_{S^2} := \partial_\phi^2 + \frac{\cos\phi}{\sin\phi}\partial_\phi + \frac{1}{\sin^2\phi}\partial_\theta^2$$

is the Laplacian on the unit sphere, $r \in [0,\infty)$, $\phi \in [0,\pi]$, and $\theta \in [0, 2\pi)$.

In the next two questions, you will be asked to perform some of the computations skipped in class in our derivation of solutions to the Schödinger equation for an electrostatic potential.

Question 2. Consider the PDE

$$\frac{d}{dx}\left((1-x^2)\frac{d\Phi}{dx}\right) + \left(\lambda - \frac{m^2}{1-x^2}\right)\Phi = 0. \tag{1}$$

Show that a solution to (1) is given by

$$\Phi(x) = (1-x^2)^{\frac{|m|}{2}}\frac{d^{|m|}P(x)}{dx^{|m|}}, \tag{2}$$

where P solves

$$(1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \lambda P = 0. \tag{3}$$

Hint: Differentiate (3) a few times and observe the resulting pattern, or use induction, to conclude that if P solves (3) then

$$(1-x^2)\frac{d^{|m|+2}P}{dx^{|m|+2}} - 2(|m|+1)x\frac{d^{|m|+1}P}{dx^{|m|+1}} + (\lambda - |m|(|m|+1))\frac{d^{|m|}P}{dx^{|m|}} = 0. \tag{4}$$

Next, let $\widetilde{\Phi}$ be defined by

$$\Phi(x) = (1-x^2)^{\frac{|m|}{2}}\widetilde{\Phi}(x), \tag{5}$$

where Φ is a solution to (1). Plugging (5) into (1), conclude that $\widetilde{\Phi}$ satisfies

$$(1-x^2)\frac{d^2\widetilde{\Phi}}{dx^2} - 2x\,(|m|+1)\,\frac{d\widetilde{\Phi}}{dx} + (\lambda - |m|(|m|+1))\,\widetilde{\Phi} = 0. \tag{6}$$

Compare (6) with (4) to obtain the result.

Question 3. Consider the PDE

$$\frac{1}{\varrho^2}\frac{d}{d\varrho}\left(\varrho^2\frac{dR}{d\varrho}\right)+\left(-\frac{1}{4}-\frac{\ell(\ell+1)}{\varrho^2}+\frac{\gamma}{\varrho}\right)R=0. \tag{7}$$

Show that (7) does not admit a non-trivial solution of the form

$$R(\varrho)=\sum_{k=0}^{\infty}a_k\varrho^k. \tag{8}$$

Hint: Plug (8) into (7) to derive

$$\begin{aligned}&-\ell(\ell+1)a_0\varrho^{-2}+\left((2-\ell(\ell+1))a_1+\gamma a_0\right)\varrho^{-1}\\&+\sum_{k=0}^{\infty}\left(\left((k+3)(k+2)-\ell(\ell+1)\right)a_{k+2}+\gamma a_{k+1}-\frac{1}{4}a_k\right)\varrho^k=0.\end{aligned} \tag{9}$$

From (9), conclude that $a_k=0$ for all k.

Question 4. Consider the following initial-boundary value problem for the wave equation in one dimension:

$$\begin{aligned}u_{tt}-c^2u_{xx}&=0, && \text{in } (0,\infty)\times(0,L),\\ u(t,0)=u(t,L)&=0, && t\geq 0,\\ u(0,x)&=g(x), && 0\leq x\leq L,\\ \partial_t u(0,x)&=h(x), && 0\leq x\leq L,\end{aligned}$$

where g and h are given and satisfy the compatibility conditions $g(0)=g(L)=0=h(0)=h(L)$. Use separation of variables to show that a formal solution to this problem is given by

$$u(t,x)=\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi ct}{L}+b_n\sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L},$$

where a_n and b_n are arbitrary coefficients.

Question 5. Consider the following initial-boundary value problem for the heat equation in one dimension:

$$\begin{aligned}u_t-u_{xx}&=0, && \text{in } (0,\infty)\times(0,L),\\ u(t,0)=u(t,L)&=0, && t\geq 0,\\ u(0,x)&=g(x), && 0\leq x\leq L,\end{aligned}$$

where g is given and satisfies the compatibility conditions $g(0)=g(L)=0$. Use separation of variables to show that a formal solution to this problem is given by

$$u(t,x)=\sum_{n=1}^{\infty}a_ne^{-\frac{n^2\pi^2}{L^2}t}\sin\frac{n\pi x}{L},$$

where the a_n's are arbitrary coefficients. What happens when $t\to\infty$? Interpret your answer physically.