HOMEWORK 1 SOLUTIONS

MATH 3120

The notation and terminology below is the same used in class.

Question 1. Review multivariable calculus, especially the chain rule in several variables and vector identities/operators.

Solution 1. Done!

Question 2. Verify whether the given function is a solution of the given PDE:

(a) $u(x,y) = y\cos x + \sin y \sin x$, $u_{xx} + u = 0$.

(b) $u(x,y) = \cos x \sin y$, $(u_{xx})^2 + (u_{yy})^2 = 0$.

Solution 2. (a) Compute $u_{xx}(x,y) = -y\cos x - \sin x \sin y = -u(x,y)$, thus u is a solution.
(b) Compute $u_{xx}(x,y) = -\cos x \sin y$, $u_{yy}(x,y) = -\cos x \sin y$, thus

$$u_{xx}(x,y))^2 + (u_{yy}(x,y))^2 = 2\cos^2 x \sin^2 y \neq 0,$$

hence u is not a solution.

Question 3. For each PDE seen as example in the first class (Laplace's equation, heat equation, wave equation, Schödinger's equation, Maxwell's equation, Euler and Navier-Stokes equations), state whether it is a scalar PDE (i.e., single PDE) or a system of PDEs, its order, and whether it is a linear or non-linear PDE. (We have not yet defined what linear vs. non-linear PDEs means, but your knowledge of ODEs should suffice for this homework set.)

Solution 3. Laplace's equation: scalar, second order, linear. Heat equation: scalar, second order (first-order in time), linear. Wave equation: scalar, second order, linear. Schödinger's equation: complex scalar, second order (first-order in time), linear. Maxwell's equation: system, first order, linear. Euler's equations: system, first order, non-linear. Navier-Stokes' equations: system, second order (first-order in time), non-linear.

Question 4. Consider a linear homogeneous PDE. Explain why any linear combination of solutions is also a solution. (Again, use your knowledge of ODE to define linearity here.)

Solution 4. Sums and multiplication by constants are preserved by linearity.

Question 5. Consider Maxwell's equations:

$$\begin{aligned} \operatorname{div} E &= \frac{\varrho}{\varepsilon_0}, \\ \operatorname{div} B &= 0, \\ \frac{\partial B}{\partial t} + \operatorname{curl} E &= 0, \\ \frac{\partial E}{\partial t} - \frac{1}{\mu_0 \varepsilon_0} \operatorname{curl} B &= -\frac{1}{\varepsilon_0} J. \end{aligned}$$

Assume that ϱ and J vanish. Show that Maxwell's equations then imply that E and B satisfy the wave equation:

$$\frac{\partial^2 E}{\partial t^2} - \frac{1}{\varepsilon_0 \mu_0} \Delta E = 0,$$

and

$$\frac{\partial^2 B}{\partial t^2} - \frac{1}{\varepsilon_0 \mu_0} \Delta B = 0.$$

Interpret your result. Can you guess what the constant $\frac{1}{\varepsilon_0 \mu_0}$ must equal to?

Solution 5. Under the assumptions, the equations become

$$\operatorname{div} E = 0, \tag{1}$$

$$\operatorname{div} B = 0, \tag{2}$$

$$\frac{\partial B}{\partial t} + \operatorname{curl} E = 0, \tag{3}$$

$$\frac{\partial E}{\partial t} - \frac{1}{\mu_0 \varepsilon_0} \operatorname{curl} B = 0. \tag{4}$$

Take the curl of (3) and note that $\operatorname{curl} \frac{\partial}{\partial t} = \frac{\partial}{\partial t} \operatorname{curl}$ to get

$$\frac{\partial}{\partial t} \operatorname{curl} B + \operatorname{curl} \operatorname{curl} E = 0.$$

But $\operatorname{curl} B = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$ by (4), thus

$$\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} + \operatorname{curl} \operatorname{curl} E = 0.$$

Recalling the following identity from multivariable calculus

$$\operatorname{curl} \operatorname{curl} f = \nabla(\operatorname{div} f) - \Delta f,$$

and using (1), we obtain the wave equation for E. The wave equation for B is similarly obtained.

The interpretation is that the electric and magnetic fields propagate in vacuum as waves. From the discussion about the wave equation in class, we conclude that $\frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ is the speed of propagation of the electromagnetic waves, which, from physics, we know to be equal to the speed of light (in vacuum).

Question 6. Consider Euler's equations:

$$\partial_t \varrho + u^i \partial_i \varrho + \varrho \partial_i u^i = 0,$$

$$\varrho(\partial_t u^j + u^i \partial_i u^j) + \nabla^j p = 0,$$

where we recall that $p = p(\varrho)$. A fluid is called *incompressible* if $\varrho = constant$, in which case we can set $\varrho = 1$. In this case, the equations describing the fluid motion are

$$\partial_t u^j + u^i \partial_i u^j + \nabla^j p = 0,$$

$$\partial_i u^i = 0,$$

which are called the *incompressible Euler equations.* For an incompressible fluid, however, the pressure is no longer given by $p = p(\varrho)$, since the pressure would then be constant, but experiments show that the pressure can vary even if the density remains (approximately) constant. Show that in the case of the incompressible Euler equations, the pressure is given as a solution to

$$\Delta p = -\partial_j u^i \partial_i u^j.$$

Solution 6. Taking the divergence of the momentum equation and using that $\partial_i u^i = 0$, we find

$$\begin{aligned} 0 &= \partial_j(\partial_t u^j + u^i \partial_i u^j + \nabla^j p) \\ &= \partial_t \partial_j u^j + \partial_j u^i \partial_i u^j + u^i \partial_i \partial_j u^j + \partial_i \partial^i p \\ &= \partial_j u^i \partial_i u^j + \partial_i \partial^i p, \end{aligned}$$

where we denoted $\partial^i := \delta^{ij}\partial_j$, with δ being the Kronecker-delta symbol defined as $\delta^{ij} = \delta_{ij} = \delta^i_j = 1$ if $i = j$ and 0 otherwise. Noting that $\partial^i \partial_i = \Delta$, we have the result.

Remark. Note that while Euler's equations in principle require functions that are only once differentiable, the above calculation assumed that the functions are in fact twice continuously differentiable.

Question 7. Consider the incompressible Euler equations (see previous question):

$$\begin{aligned} \partial_t u^j + u^i \partial_i u^j + \nabla^j p &= 0, \\ \partial_i u^i &= 0. \end{aligned}$$

The *vorticity* ω of the fluid is defined as

$$\omega := \operatorname{curl} u.$$

The vorticity is an important physical quantity; it measures, as the name suggests, "eddies" in the fluid. It is, therefore, important to know how it changes in time and space (i.e., what the dynamics of the vorticity is). Show that ω satisfies the following PDE:

$$\partial_t \omega + \nabla_u \omega - \nabla_\omega u = 0.$$

Above, the operators ∇_u and ∇_ω are defined as follows. For any vector field X, ∇_X is a short hand notation for $X \cdot \nabla$, i.e.,

$$\nabla_X := X \cdot \nabla,$$

where we recall that $X \cdot \nabla$ has been defined in class as

$$X \cdot \nabla = X^i \partial_i.$$

Solution 7. Denoting by $|\cdot|$ the norm in $\mathbb{R}^3$, observe the following identity:

$$\frac{1}{2}\nabla^i |u|^2 = \frac{1}{2}\nabla^i (u^\ell u_\ell) = u^\ell \partial^i u_\ell = u^\ell \partial_\ell u^i + (u^\ell \partial^i u_\ell - u^\ell \partial_\ell u^i),$$

where ∂^i is as in the last question. Next, compute

$$\begin{aligned} (u \times \omega)^i &= \epsilon^{ijk} u_j \omega_k = \epsilon^{ijk} u_j \epsilon_k{}^{\ell n} \partial_\ell u_n \\ &= (\delta^{i\ell}\delta^{jn} - \delta^{j\ell}\delta^{in}) u_j \partial_\ell u_n \\ &= u^n \partial^i u_n - u^\ell \partial_\ell u^i, \end{aligned}$$

where we used the identity

$$\epsilon^{ijk}\epsilon_{k\ell n} = \epsilon^{kij}\epsilon_{k\ell n} = \delta^i_\ell \delta^j_n - \delta^j_\ell \delta^i_n,$$

which can be verified directly. From the foregoing we conclude that

$$\nabla_u u = \frac{1}{2}\nabla |u|^2 - u \times \omega,$$

which implies

$$\operatorname{curl} \nabla_u u = -\operatorname{curl}(u \times \omega).$$

Let us compute the RHS:

$$\begin{aligned}
(\operatorname{curl}(u\times\omega))^i &= \epsilon^{ijk}\partial_j \omega_k = \epsilon^{ijk}\partial_j(\epsilon_k{}^{\ell n}\partial_\ell u_n) \\
&= \epsilon^{ijk}\epsilon_k{}^{\ell n}\partial_j u_\ell \omega_n + \epsilon^{ijk}\epsilon_k{}^{\ell n} u_\ell \partial_j \omega_n \\
&= (\delta^{i\ell}\delta^{jn} - \delta^{j\ell}\delta^{in})\partial_j u_\ell \omega_n + (\delta^{i\ell}\delta^{jn} - \delta^{j\ell}\delta^{in}) u_\ell \partial_j \omega_n \\
&= \partial^n u^i \omega_n - \underbrace{\partial_\ell u^\ell}_{=0} \omega^i + u^i \underbrace{\partial_n \omega^n}_{=0} - u^j \partial_j \omega^i \\
&= (\nabla_\omega u)^i - (\nabla_u \omega)^i,
\end{aligned}$$

which implies the result.