CONVERGENCE OF FOURIER SERIES

MATH 3120

Here we collect some useful results about convergence of Fourier series. Their proofs can be found in many textbooks, e.g., [1, 2].

We will denote by $f(x^+)$ and $f(x^-)$ the right and left values of a function f at x, defined by

$$f(x^+) = \lim_{h\to 0^+} f(x+h),$$

and

$$f(x^-) = \lim_{h\to 0^-} f(x+h).$$

If f is continuous at x, we have that $f(x^+) = f(x^-)$, but in general these values need not to be equal. For instance, let

$$f(x) = \begin{cases} -1, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

Then $f(0^+) = 1$ and $f(0^-) = -1$.

As done in class, the Fourier series of a function f will be written as

$$F.S.\{f\}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}).$$

Theorem 1. *Let f be a piecewise C^1 function defined on $[-L, L]$. Then, for any $x \in (-L, L)$,*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}) = \frac{1}{2}(f(x^+) + f(x^-))$$

For $x = \pm L$, the series converges to $\frac{1}{2}(f(-L^+) + f(L^-))$.

Thus, the Fourier series of (a piecewise C^1 function) f at x converges to $f(x)$ if f is continuous at x.

Next, we consider differentiation and integration of Fourier series.

Theorem 2. *Let f be continuous on $[-L, L]$. Suppose that $f(-L) = f(L)$, and that f is piecewise C^2. Then, the Fourier series of f' can be obtained from that of f by differentiation term-by-term. I.e., writing*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}),$$

then

$$F.S.\{f'\}(x) = \sum_{n=1}^{\infty} (a_n (\cos\frac{n\pi x}{L})' + b_n (\sin\frac{n\pi x}{L})'),$$

In particular, if $f'(x) = F.S.\{f'\}(x)$, then

$$f'(x) = \sum_{n=1}^{\infty}(-a_n \frac{n\pi}{L} \sin \frac{n\pi x}{L} + b_n \frac{n\pi}{L} \cos \frac{n\pi x}{L}).$$

The assumption that $f(-L) = f(L)$ means that we could think of f as the restriction to $[-L, L]$ of a continuous $2L$-periodic function.

Finally,

Theorem 3. *Let f be a continuous function on $[-L, L]$ with Fourier series*

$$F.S.\{f\}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}).$$

Then, for any $x \in (-L, L)$, we have

$$\int_{-L}^{x} f(s)\, ds = \int_{-L}^{x} \frac{a_0}{2}\, ds + \sum_{n=1}^{\infty}(a_n \int_{-L}^{x} \cos \frac{n\pi s}{L}\, ds + b_n \int_{-L}^{x} \sin \frac{n\pi s}{L}\, ds).$$

We now illustrate how one can use finitely many terms of the Fourier series to approximate a function. I.e., instead of taking an infinite sum in the Fourier series, we consider only the first N terms:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{N}(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}).$$

Figure 1 shows the function $|x|$ for $-\pi \leq x \leq \pi$ and its Fourier series with $N = 2$ and $N = 4$ terms.

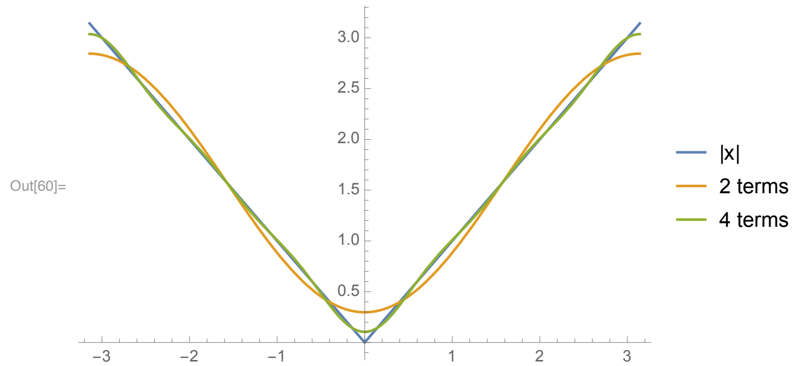

FIGURE 1. Graph of the function $|x|$ and its corresponding Fourier series consisting only of the first two and four terms.

Figure 2 shows the function

$$f(x) = \begin{cases} -1, & -\pi \leq x < 0, \\ 1, & 0 \leq x \leq 0. \end{cases}$$

and its Fourier series with $N = 6$ and $N = 18$ terms.

Figure 3 shows the function $\frac{x}{2}$ for $-\pi \leq x \leq \pi$ and extended to a 2π-periodic function, and its Fourier series with $N = 4$ and $N = 8$ terms.

In all these cases, the larger the N, the better the approximation, as expected.

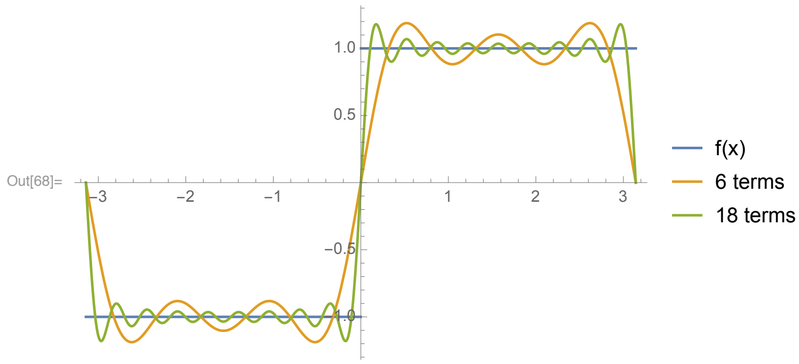

FIGURE 2. Graph of the function f and its corresponding Fourier series consisting only of the first six and 18 terms.

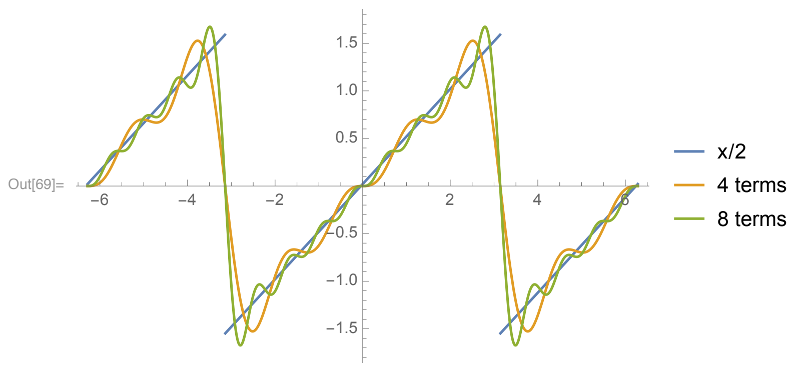

FIGURE 3. Graph of the function $\frac{x}{2}$ (2π-periodic), and its corresponding Fourier series consisting only of the first four and eight terms.

REFERENCES

[1] T. Myint-U, *Partial differential equations of mathematical physics*, Elsevier Science Ltd, 1980.
[2] W. A. Strauss, *Partial differential equations: An introduction 2nd edition*, Wiley, 2007.