VANDERBILT UNIVERSITY

MATH 3120 – INTRO DO PDES

Test 1

NAME: Solutions.

Directions. This exam contains five questions. Make sure you clearly indicate the pages where your solutions are written. Answers without justification will receive little or no credit. Write clearly, legibly, and in a logical fashion. Make precise statements (for instance, write an equal sign if two expressions are equal; say that one expression is a consequence of another when this is the case, etc). Notice that different questions may be worth different amounts of points.

If you do not understand a question, or think that some problem is ambiguous, missing information, or incorrectly stated, write how you interpret the problem and solve it accordingly.

Question	**Points**
1 (15 pts)	
2 (20 pts)	
3 (20 pts)	
4 (25 pts)	
5 (20 pts)	
TOTAL	

Question 1. (15 pts) For each PDE below, identify the unknown function and state the independent variables. State the order of the PDE. State if the PDE is homogeneous or non-homogeneous, linear or non-linear. (You are asked to state whether the PDE is homogeneous/non-homogeneous and linear/non-linear, not to prove your statement. In particular, you **need not** to write the PDE in the form $F(x, u, \dots, D^m u) = 0$ or identify the function F.)

(a) $u_{tt} - u_{xx} = f$.

(b) $\cos(t) u_t + x^2 u_x + y^2 u_y = e^{-x^2 - y^2} \sin(t)$.

(c) $u_{xx} + u_{yy} = e^u$.

Solution 1. (a) Unknown: u. Independent variables: x, t. Order: second. Non-homogeneous PDE. Linear PDE.

(b) Unknown: u. Independent variables: x, y, t. Order: first. Non-homogeneous PDE. Linear PDE.

(c) Unknown: u. Independent variables: x, y. Order: second. Homogeneous PDE. Non-linear PDE.

Question 2. (20 pts) Consider the initial-value problem:

$$xu_x + yu_y = 4u, \ -\infty < x < \infty, \ -\infty < y < \infty,$$
$$u = 1 \text{ on the circle } x^2 + y^2 = 1.$$

(a) Identify the PDE and the initial condition.

(b) Find a solution $u = u(x, y)$ for the initial-value problem.

(c) Sketch the projection of the characteristic curves on the xy-plane (i.e., sketch the projected characteristics).

(d) Is the solution you found in (b) a local or global solution. Is it unique?

Solution 2. (a) PDE: $xu_x + yu_y = 4u$. Initial condition: $u = 1$ on the circle $x^2 + y^2 = 1$.

(b) Parametrize the initial condition as

$$\Gamma(s) = (x_0(s), y_0(s), u_0(s)) = (\cos s, \sin s, 1), \ 0 \le s < 2\pi.$$

The system of characteristic equations is

$$\dot{x} = x,$$
$$\dot{y} = y,$$
$$\dot{u} = 4u.$$

The solution is

$$x(t, s) = e^t \cos s, y(t, x) = e^t \sin s, \ u(t, s) = e^{4t}.$$

Compute

$$x^2 + y^2 = (e^t \cos s)^2 + (e^t \sin s)^2 = e^{2t},$$

hence

$$u(x, y) = (x^2 + y^2)^2.$$

(c) We have $y/x = \tan s$, hence the characteristics are straight lines through the origin, see Figure 1.

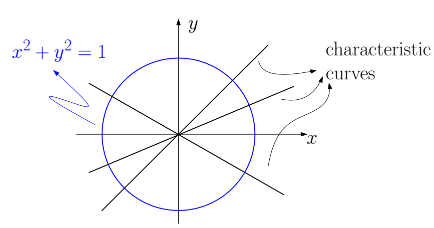

FIGURE 1. Projected characteristics of problem 2c.

(d) Compute

$$J(0, s) = \det \begin{bmatrix} \partial_t x(0, s) & \partial_s x(0, s) \\ \partial_t y(0, s) & \partial_s y(0, s) \end{bmatrix} = \det \begin{bmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{bmatrix} = 1.$$

Thus, the transversality condition holds and the solution is unique in a neighborhood of Γ.

Question 3. (20 pts) Consider the following initial-value problem:

$$u_t + uu_x = 0 \text{ in } (-\infty, \infty) \times (0, \infty)$$
$$u(x, 0) = h(x), \; -\infty < x < \infty,$$

where h is a given function.

(a) Verify that

$$u(x, t) = h(x - tu(x, t))$$

gives an implicit solution for the initial-value problem.

(b) Show that there exist initial conditions h for which the solution u blows-up (i.e., forms a shock) at a certain finite time t_*, and find a formula for t_*.

(c) Do there exist initial conditions h for which no shock occurs?

Solution 3. (a) Clearly $u(x, 0) = h(x)$. Differentiating $u(x, t) = h(x - tu(x, t))$ with respect to t gives

$$u_t = h'(x - tu)(-u - tu_t),$$

and differentiating with respect to x produces.

$$u_x = h'(x - tu)(1 - tu_x).$$

Hence

$$u_t + uu_x = -th'(x - tu)(u_t + uu_x),$$

or

$$(1 + th'(x - tu))(u_t + uu_x) = 0.$$

Hence $u_t + uu_x = 0$ as long as $1 + th'(x - tu) \neq 0$. Since $1 + th'(x - tu) = 0$ is the condition for shocks, we have shown that the given formula defines an implicit solution as long as shocks do not occur.

(b) From the formula $u_x = h'(x - tu)(1 - tu_x)$ computed above we obtain

$$u_x = \frac{h'}{1 + th'}.$$

Hence, for h such that $h'(s) < 0$ for some $s \in \mathbb{R}$, we obtain that u_x blows-up at time

$$t_* = -\frac{1}{h'(s)}.$$

(c) If $h'(s) > 0$ for all s, then no blow-up occurs (recall that $t \geq 0$). From the formula $(1 + th'(x - tu))(u_t + uu_x) = 0$ derived in (a), we also see that u remains a solution for all $t \geq 0$.

Question 4. (25 pts) Consider the quasi-linear first order PDE:

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u), \tag{1}$$

where a, b, and c are given functions. Assume that an initial condition u_0 is given on a curve Γ in $\mathbb{R}^2$, i.e.,

$$u = u_0 \text{ on } \Gamma. \tag{2}$$

(a) Write the system of characteristic equations for (1). (You are **not** asked to derive the system of characteristic equations, only to state it.)

(b) State conditions on a, b, c, and Γ that guarantee that equation (1) with initial condition (2) has a unique solution in some neighborhood of Γ. Draw a picture illustrating the conditions you stated and explain why they guarantee the existence and uniqueness of a solution. (You are **not** asked to give a formal proof of your statement. Rather, you have to explain in words and with the help of your picture why the stated result is true.)

(c) Explain why existence and/or uniqueness can fail when the conditions you stated in (b) are not fulfilled. Draw a picture illustrating your argument. (Again, you are **not** asked to give a formal proof, only to explain in words.)

Solution 4. (a) Parametrize the curve Γ by $(x_0(s), y_0(s))$, where $s \in (\alpha, \beta)$ (with possibly $\alpha = -\infty$ or $\beta = \infty$). The system of characteristic equations is

$$\begin{aligned} \dot{x} &= a(x,y,u), \\ \dot{y} &= b(x,y,u), \\ \dot{u} &= c(x,y,u), \end{aligned}$$

with initial conditions

$$x(0,s) = x_0(s),\ y(0,s) = y_0(s),\ u(0,s) = u_0(s),$$

where $u_0(s) = u_0(x_0(s), y_0(s))$.

(b) Assume that Γ is a smooth curve and that a, b, and c are smooth functions in a neighborhood of

$$\Gamma(s) = (x_0(s), y_0(s), u_0(s)),\ s \in (\alpha, \beta).$$

(As in class, we abuse the notation, denoting by Γ the initial curve in $\mathbb{R}^2$ and by $\Gamma(s)$ the parametrized curve in $\mathbb{R}^3$ that includes the initial condition u_0, see Figure 2.) Suppose that there exist a $s_0 \in (\alpha, \beta)$ and a $\delta > 0$ satisfying $(s_0 - 2\delta, s_0 + 2\delta) \subset (\alpha, \beta)$ and such that for all $s \in (s_0 - 2\delta, s_0 + 2\delta)$, it holds that

$$\det \begin{bmatrix} a(x_0(s), y_0(s), u_0(s)) & \partial_s x_0(s) \\ b(x_0(s), y_0(s), u_0(s)) & \partial_s y_0(s) \end{bmatrix} \neq 0. \tag{3}$$

Then, there exists a unique solution u to (1) satisfying (2) in a neighborhood of a portion of Γ containing $(x_0(s), y_0(s))$ for $s \in (s_0 - \delta, s_0 + \delta)$.

Condition (3) is the transversality condition. It guarantees that the (projected) characteristic curves will be transversal to Γ for $s \in (s_0 - 2\delta, s_0 + 2\delta)$. Hence, the characteristic curves starting on $\Gamma(s)$ will exist for a small time that is uniform in $s \in (s_0 - \delta, s_0 + \delta)$. Furthermore, for such a small time, the characteristic curves will be non-intersecting and their projection onto the xy-plane will cover a neighborhood of (a portion of) Γ. Hence, their union will form a smooth parametrize surface that yields the solution u.

Geometrically, what is happening is illustrated in Figure 2. The meaning of (3) is that the two-component vectors

$$(\partial_s x_0(s), \partial_s y_0(s)) \text{ and } (a(x_0(s), y_0(s), u_0(s)), b(x_0(s), y_0(s), u_0(s)))$$

are linearly independent, thus they are not a multiple of each other. This implies that the characteristic curve tangent to the three-component vector

$$(a(x_0(s), y_0(s), u_0(s)), b(x_0(s), y_0(s), u_0(s)), c(x_0(s), y_0(s), u_0(s)))$$

must be transverse to $\Gamma(s)$, and leave $\Gamma(s)$ in the x and y directions (note that $(\partial_s x_0(s), \partial_s y_0(s), \partial_s u_0(s))$ is tangent to curve $\Gamma(s)$). Therefore, the union of the characteristic curves forms a surface that corresponds to u.

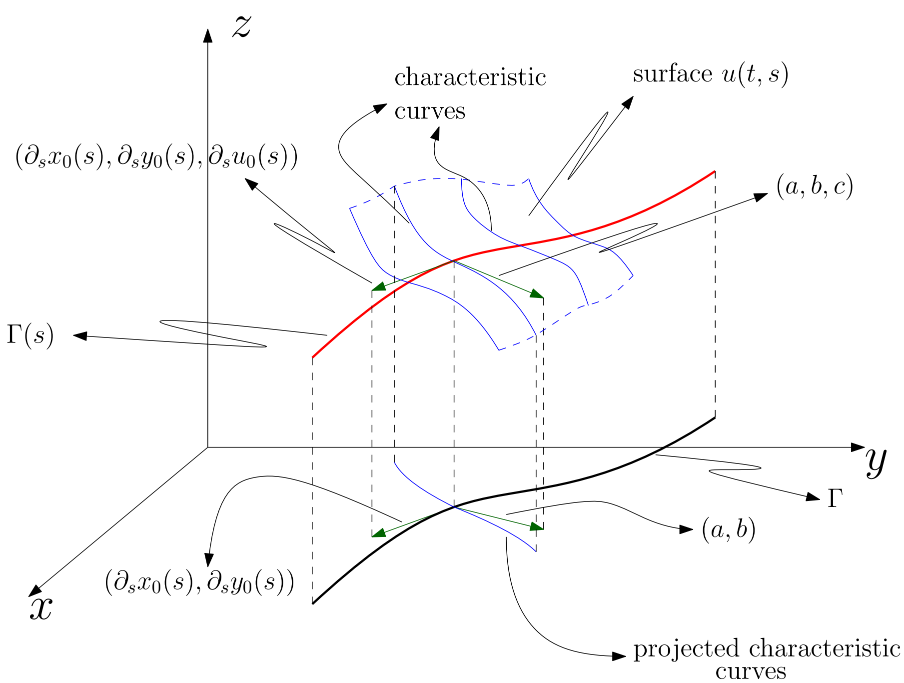

FIGURE 2. Geometric interpretation of the existence and uniqueness result when the transversality condition holds.

(c) If the transversality condition fails for an interval of s values, then either the characteristic curves starting at that interval coincide with the initial curve or they do not. In the latter case, this means that the tangent vectors to the characteristic curves cannot be tangent to any parametrized surface representing a solution u, as the union of the characteristic curves do not form the portion of a graph in the x, y variables.

In the second case, we can choose any curve $\widetilde{\Gamma}$ transversal to Γ and prescribe initial data on $\widetilde{\Gamma}$. We obtain a solution that also satisfy (2) since now Γ is a (projected) characteristic for the system, hence contained in the solution constructed for the initial condition $\widetilde{\Gamma}$.

The cases when the transversality condition fails are illustrated in Figures 3 and 4. The failure of (3) means that the vectors

$$(\partial_s x_0(s), \partial_s y_0(s)) \text{ and } (a(x_0(s), y_0(s), u_0(s)), b(x_0(s), y_0(s), u_0(s)))$$

are linearly dependent, thus they are a multiple of each other. Hence, the vectors

$$(\partial_s x_0(s), \partial_s y_0(s), \partial_s u_0(s))$$

and

$$(a(x_0(s), y_0(s), u_0(s)), b(x_0(s), y_0(s), u_0(s)), c(x_0(s), y_0(s), u_0(s))),$$

are either a multiple of each other (Figure 4) or fail to be a multiple of each other only with respect to the third component (Figure 3). In the later case, the characteristic curves are "above" or "below" $\Gamma(s)$. Hence, their union forms a surface containing $\Gamma(s)$ that is "vertical." But such a vertical surface cannot be the graph of a function, hence no solution exists. In the case when the above three-component vectors are a multiple of each other (Figure 4), the chacteristic curves coincide with $\Gamma(s)$. In particular, they will be characteristic curves of the solution with initial conditions on $\widetilde{\Gamma}$, providing a solution $\widetilde{u}$. But since we can make infinitely many choices for $\widetilde{\Gamma}$, solutions are not unique.

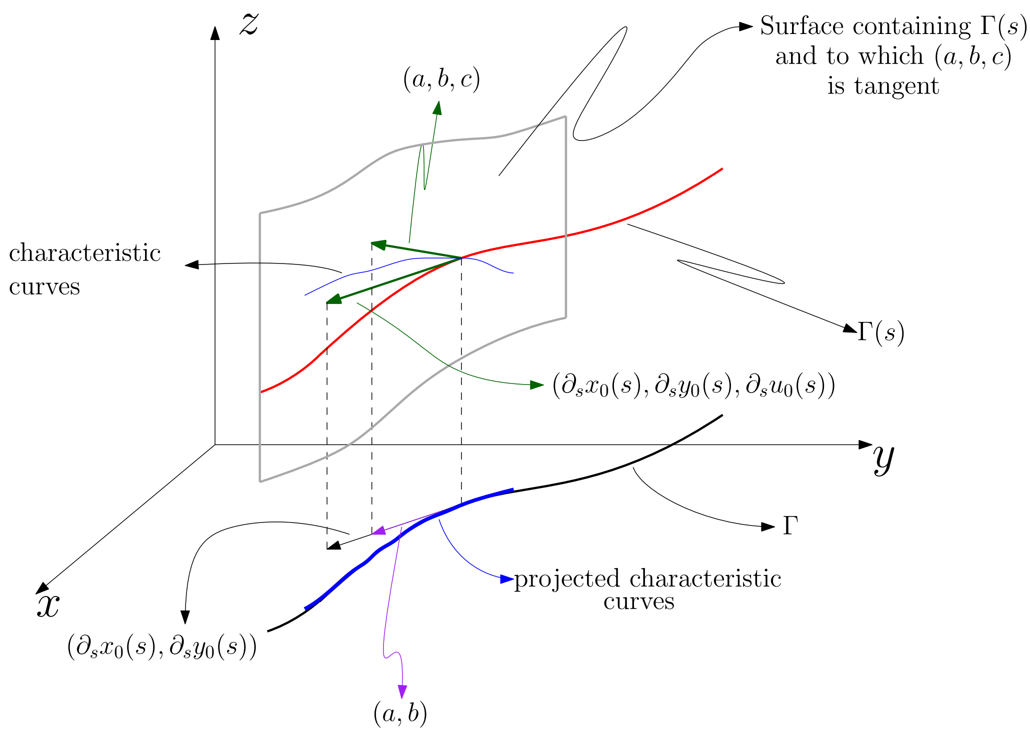

FIGURE 3. Geometric interpretation of the non-existence of solutions.

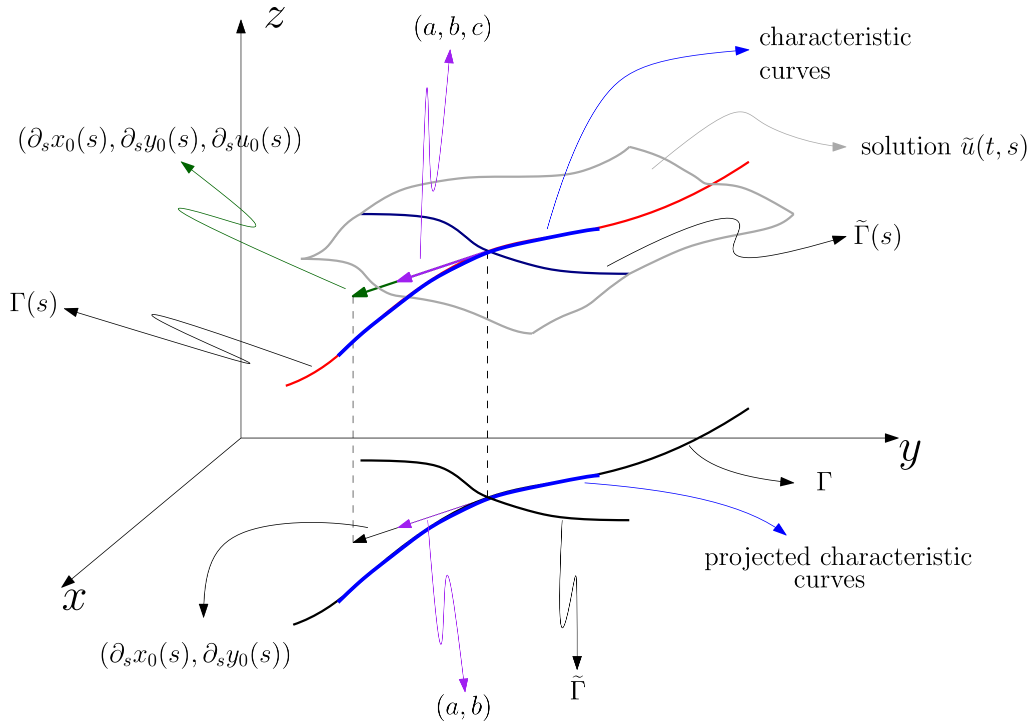

Figure 4. Geometric interpretation of the non-uniqueness of solutions.

Question 5. (20 pts) Consider the following initial-value problem for the wave equation in one dimension:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \text{ in } (-\infty, \infty) \times (0, \infty), \\ u(x, 0) &= f(x), \\ u_t(x, 0) &= g(x), \end{aligned} \tag{4}$$

(a) Solve (4) when $f(x) = x^2$ and $g(x) = 0$.

(b) Assume now that $c = 1$ and

$$f(x) = \begin{cases} 1, & -2 \leq x \leq 0 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$g(x) = \begin{cases} -1, & -1 \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Draw a diagram in the (x, t)-plane indicating the different regions where the solution is influenced by the initial condition f and g and the regions where the solution is identically zero. (This is similar to what was done in class. You do **not** have to find u.)

Solution 5. (a) By D'Alembert's formula:

$$u(x, t) = \frac{(x + ct)^2 + (x - ct)^2}{2}.$$

(b) The regions are summarized in the Figure 5.

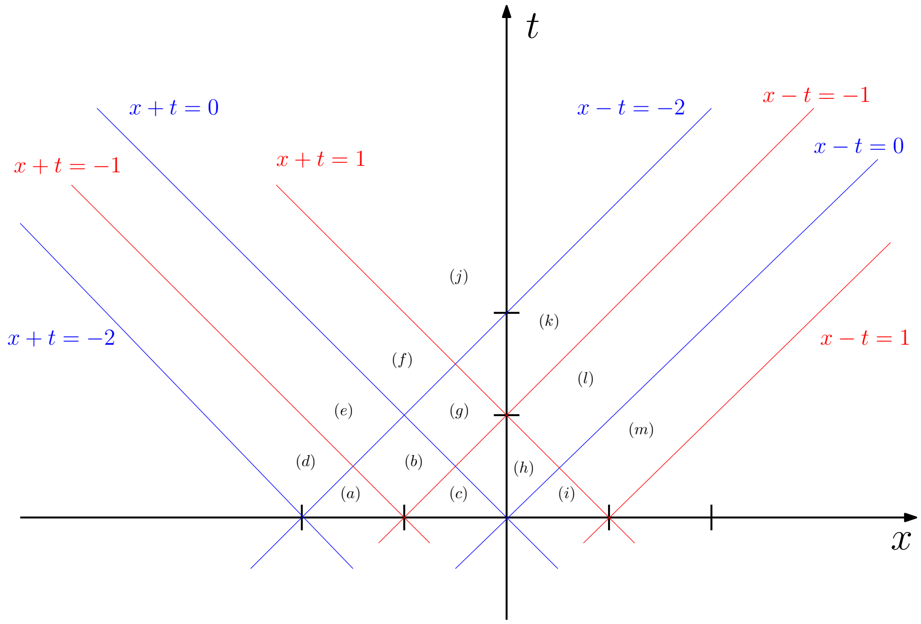

FIGURE 5. Problem 5b.