VANDERBILT UNIVERSITY

MATH 3120 – INTRO DO PDES

HW 4 - Solutions

Question 1. Use separation of variables to solve the following initial-boundary value problem for the wave equation (the only difference from what was done in class is the boundary condition):

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 && \text{in } (0,L)\times(0,\infty), \\ u(x,0) &= f(x) && 0 \le x \le L, \\ u_t(x,0) &= g(x) && 0 \le x \le L, \\ u_x(0,t) &= 0 && t \ge 0, \\ u_x(L,t) &= 0 && t \ge 0. \end{aligned}$$

Solution 1. The separation of variables is done as in class. The difference is that now for the equation

$$X'' + \lambda X = 0,$$

we use the boundary conditions

$$X'(0) = X'(L) = 0.$$

We then find $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 0, 1, 2, \dots$ and

$$X_n(x) = \cos\left(\frac{n\pi x}{L}\right), \; n = 0, 1, 2, \dots .$$

Differently than what was done in class, here $n = 0$ is included. Using λ_n in the T equation now produces

$$\begin{aligned} T_0(t) &= a_0 + b_0 t, \\ T_n(t) &= a_n \cos\left(\frac{n\pi c t}{L}\right) + b_n \sin\left(\frac{n\pi c t}{L}\right), \; n = 1, 2, 3, \dots \end{aligned}$$

The solution T_0 corresponds to using λ_0 in the equation for T. We obtain, after redefining the coefficients a_0 and b_0,

$$u(x,t) = \frac{a_0 + b_0 t}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi c t}{L}\right) + b_n \sin\left(\frac{n\pi c t}{L}\right) \right) \cos\left(\frac{n\pi x}{L}\right),$$

where a_n and b_n are given by the familiar formulas for Fourier coefficients on $[0, L]$.

Question 2. Show that the solution you found in problem 1 can be written as a superposition of a forward and a backward wave.

Solution 2. We will write u as $u = u_1 + u_2 + \frac{a_0}{2}$, where u_1 is a forward wave, u_2 a backward wave, and $\frac{a_0}{2}$, being a constant, can be thought of as either a forward or backward wave.

Using the trigonometric identities

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$$

and

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

we find

$$u_1(x,t) = -\frac{b_0}{4c}(x-ct) + \sum_{n=1}^{\infty}\left(\frac{a_n}{2}\cos\left(\frac{n\pi(x-ct)}{L}\right) - \frac{b_n}{2}\sin\left(\frac{n\pi(x-ct)}{L}\right)\right),$$

and

$$u_2(x,t) = \frac{b_0}{4c}(x+ct) + \sum_{n=1}^{\infty}\left(\frac{a_n}{2}\cos\left(\frac{n\pi(x+ct)}{L}\right) + \frac{b_n}{2}\sin\left(\frac{n\pi(x+ct)}{L}\right)\right).$$

Question 3. Solve problem 1 with $c = 1$, $L = \pi$, $f(x) = \sin^3 x$, and $g(x) = \sin(2x)$.

Solution 3. The only coefficients that do not vanish are $a_1 = -1/4$, $a_3 = 3/4$, and $b_2 = 1/2$, leading to

$$u(x,t) = -\frac{1}{4}\sin(3x)\cos(3t) + \frac{3}{4}\sin x\cos t + \frac{1}{2}\sin(2x)\sin(2t).$$

Question 4. Use separation of variables to solve the following initial-boundary value problem for the heat equation:

$$\begin{aligned} u_t - ku_{xx} &= 0 && \text{in } (0,L)\times(0,\infty),\\ u(x,0) &= f(x) && 0 \le x \le L,\\ u(0,t) &= 0 && t \ge 0,\\ u(L,t) &= 0 && t \ge 0. \end{aligned}$$

Interpret your result.

Solution 4. Separating variables $u(x,t) = X(x)T(t)$ we find

$$X'' + \lambda X = 0,$$

and

$$T' + \lambda k T = 0.$$

Using the boundary conditions $X(0) = X(L) = 0$ we find $\lambda_n = (n\pi/L)^2$ and

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right),\ n = 1,2,3,\ldots$$

Using λ_n in the equation for T gives

$$T_n(t) = e^{-\frac{n^2\pi^2}{L^2}kt},\ n = 1,2,3,\ldots.$$

Hence

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2}{L^2}kt},$$

where

$$b_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx.$$

Since $e^{-\frac{n^2\pi^2}{L^2}kt} \to 0$ as $t \to \infty$, we see that $u \to 0$ as $t \to \infty$. This means that the temperature will eventually reach zero, as it should be for an insulated rod kept at zero temperature at its endpoints.

Question 5. Solve problem 4 with $k = 17$, $L = \pi$, and

$$f(x) = \begin{cases} 0, & 0 \le x \le \frac{\pi}{2}, \\ 2, & \frac{\pi}{2} < x \le \pi. \end{cases}$$

Discuss the convergence of the solution you found.

Solution 5. Computing

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx \\ &= \frac{4}{\pi n}\left(\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right). \end{aligned}$$

Thus

$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right)\sin(nx)\,e^{-17n^2t}.$$

Since

$$\left|\frac{1}{n}\left(\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right)e^{-17n^2t}\right| \le \frac{2}{n}e^{-17n^2t}$$

and the series

$$\sum_{n=1}^{\infty}\frac{2}{n}e^{-17n^2t}$$

converges for each $t > 0$, we conclude that the series for u does converge for each $t > 0$.