So far we have not discussed the convergence of the series giving the solution $u(x,t)$ that we constructed. Solutions at this stage will be called formal solutions. In other words, by a <u>formal solution</u>(*) to a PDE (or initial/boundary value problem), we mean an expression such that, if we ignore issues of convergence, continuity, existence of derivatives, etc, and carry out term-by-term differentiations and substitutions, then the expression satisfies the PDE.

When a formal solution converges to a C^k function (where k is the order of the equation, $k=2$ for the wave equation) then the formal solution is an actual solution, i.e., a classical solution. If it converges to a piece-wise C^k function then the formal solution is a generalized solution.

(*) think of a formal solution as a candidate for a solution.

Formal Aspects of Fourier series

We will be working on $[-L, L]$ rather than $[0, L]$. The connection to the Fourier series of sines and cosines studied earlier (defined on $[0, L]$) will be made later on.

We will make use of the following:

$$\int_{-L}^{L} \sin\frac{n\pi x}{L} \cos\frac{n\pi x}{L}\, dx = 0, \quad \int_{-L}^{L} \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

$$\int_{-L}^{L} \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx = \begin{cases} 0 & m \neq n \\ L & m = n \neq 0 \\ 2L & m = n = 0 \end{cases}, \quad \text{where } m \text{ and } n \text{ are integers.}$$

In general, we say that two functions f, g are <u>orthogonal</u> if $\langle f, g \rangle = 0$. The above equalities thus say that sin and cos $\frac{n\pi x}{L}$ are always orthogonal when $n \neq m$, and the interval is $[-L, L]$.

Let f be defined on $[-L, L]$. Our goal is to write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right), \quad a_n, b_n \text{ real numbers},$$

We write the coefficient a_0 separately and differently because of the factor 2 in $\langle \cos(\frac{n\pi x}{L}), \cos(\frac{n\pi x}{L}) \rangle$ for $n = 0$.

Using the orthogonality of sin and cos just stated, we can immediately compute the coefficients as we did before. They are given by:

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n = 0, 1, 2, 3, \ldots \qquad , \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3, \ldots$$

This, of course, assumes that the series converges and the limit is f. We don't know if that is the case, but we are led to the following.

Def. Let f be a piece-wise continuous function defined on $(-L, L)$.
(or $[-L, L]$)

The <u>Fourier series</u> of f, denoted F.S.$\{f\}$, or F.S.$\{f\}(x)$, is the series

$$F.S.\{f\}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where a_n and b_n are given by the above formulas, and are called <u>Fourier coefficients</u>

<u>Important remark</u>: The Fourier series F.S.$\{f\}$ is a series constructed out of f, but we are not claiming that $f = $ F.S.$\{f\}$. In fact, we are not even claiming that F.S.$\{f\}$ converges (Although we do want to find conditions that guarantee that F.S.$\{f\}$ converges and $f = $ F.S.$\{f\}$.)

Ex: Find the Fourier series of $f(x) = \begin{cases} -1, & -\pi \leq x < 0 \\ 1, & 0 \leq x \leq \pi \end{cases}$

We compute: $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = 0$ since f is odd and cos even.
(except for $x=0$)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \underset{\text{sin is odd}}{=} \frac{2}{\pi}\int_0^{\pi} \overbrace{f(x)}^{=1}\sin(nx)\,dx = \frac{2}{\pi}\left(-\frac{\cos nx}{n}\right)\Bigg|_0^{\pi}$$

$$= \frac{2}{\pi}\left(\frac{1}{n} - \frac{(-1)^n}{n}\right) = \begin{cases} 0 & n \text{ even} \\ \frac{4}{\pi n} & n \text{ odd.} \end{cases}$$

Thus

$$F.S\{f\}(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n}\right)\sin(nx)$$

$$= \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \cdots\right)$$

Remark:

$f(0) = 1$ but

$F.S.\{f\}(0) = 0$

Ex: Find the Fourier series for $f(x) = |x|$, $-1 \le x \le 1$.

Compute $b_n = \int_{-1}^{1} f(x) \sin(n\pi x)\, dx = 0$ since f is even

$$a_0 = \int_{-1}^{1} f(x)\, dx = 2\int_0^1 x\, dx = 1.$$

$$a_n = \int_{-1}^{1} f(x) \cos(n\pi x)\, dx = 2\int_0^1 x \cos(n\pi x)\, dx = \frac{2}{\pi^2 n^2}\left((-1)^n - 1\right), \quad n = 1, 2, \ldots$$

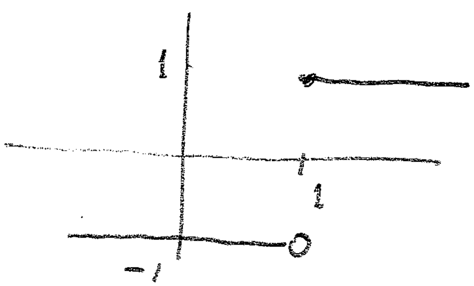
Thus

$$F.S\{f\}(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left((-1)^n - 1\right)\cos(n\pi x)$$

$$= \frac{1}{2} - \frac{4}{\pi^2}\left(\cos(\pi x) + \frac{1}{9}\cos(3\pi x) + \frac{1}{25}\cos(25\pi x) + \ldots\right)$$

Convergence of Fourier series

<u>Notation</u> We denote by $f(x^+)$ and $f(x^-)$ the right and left values of f at x, defined by $f(x^+) = \lim_{h\to 0^+} f(x+h)$, and $f(x^-) = \lim_{h \to 0^-} f(x+h)$. We have $f(x^+) = f(x^-) = f(x)$ when f is continuous, but $f(x^+)$ and $f(x^-)$ can be different otherwise



$$f(1^+) = 1, \quad f(1^-) = -1.$$

We will now state results about convergence, differentiation, and integration of Fourier series.

Theorem Let f be a piecewise C^1 function defined on $[-L, L]$. Then, for any $x \in (-L, L)$:

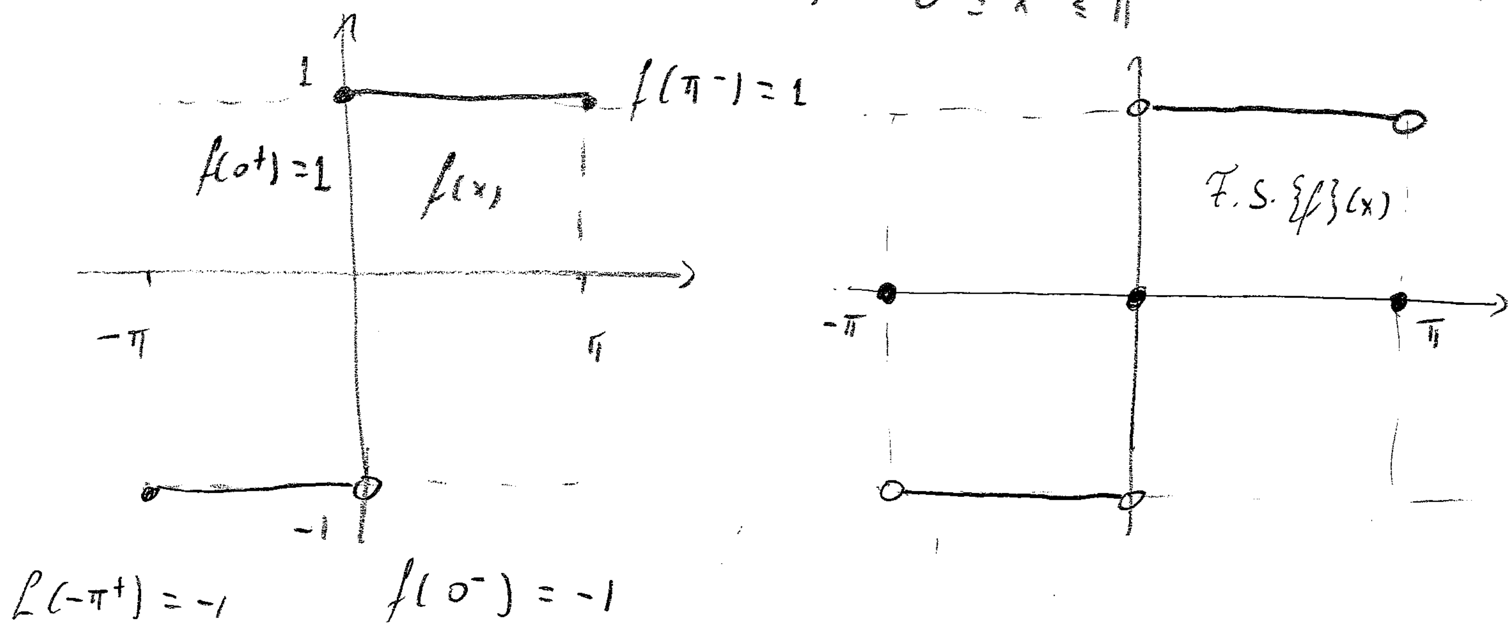
$$\underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)}_{= F.S.\{f\}} = \frac{1}{2}\left(f(x^+) + f(x^-) \right),$$

where a_n and b_n are as before. For $x = \pm L$, the series converges to $\frac{1}{2}\left(f(-L^+) + f(L^-) \right)$.

Therefore, if f is piecewise C^1, we have $F.S.\{f\}(x) = f(x)$ if f is continuous at x and $F.S.\{f\}(x) = \frac{1}{2}\left(f(x^+) + f(x^-)\right)$ otherwise. In particular, if f is piecewise C^1 and continuous, we have:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L} x\right) + b_n \sin\left(\frac{n\pi}{L} x\right) \right)$$

Ex: We graph $f(x) = \begin{cases} -1, & -\pi \le x < 0 \\ 1, & 0 \le x < \pi \end{cases}$ and $F.S.\{f\}(x)$ below.



Note the difference between f and $F.S.\{f\}$ at the points of discontinuity and at the endpoints.

Ex: Since $|x|$ is continuous, we have

$$|x| = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left((-1)^n - 1\right)\cos(n\pi x)$$

Next, we consider differentiation and integration of Fourier series term-by-term, in the spirit of what we did when we solved the wave equation.

<u>Theorem</u> Let f be continuous on $[-L,L]$. Suppose that $f(-L)=f(L)$, and that f is piecewise C^2. Then, the Fourier series of f' can be obtained from that of f by differentiation term-by-term. I.e., if

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right),$$

then

$$f'(x) = \sum_{n=1}^{\infty} \left(a_n \underbrace{\left(\cos\left(\frac{n\pi x}{L}\right)\right)'}_{= -\frac{n\pi}{L}\sin\left(\frac{n\pi x}{L}\right)} + b_n \underbrace{\left(\sin\left(\frac{n\pi x}{L}\right)\right)'}_{= \frac{n\pi}{L}\cos\left(\frac{n\pi}{L}x\right)} \right)$$

We now state a similar result for integration.

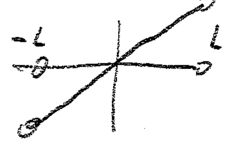
Theorem Let f be a piecewise continuous function on $[-L,L]$ and assume that

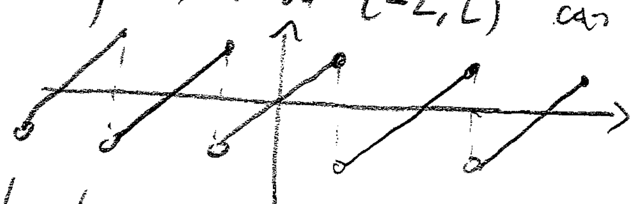
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

Then, for any $x \in [-L,L]$, we have

$$\int_{-L}^{x} f(t)\,dt = \int_{-L}^{x}\frac{a_0}{2}\,dt + \sum_{n=1}^{\infty}\left(a_n\int_{-L}^{x}\cos\left(\frac{n\pi t}{L}\right)dt + b_n\int_{-L}^{x}\sin\left(\frac{n\pi t}{L}\right)dt\right)$$

The case of periodic functions Suppose that f is defined on $\mathbb{R}$ and has period $2L$. ($f(x+2L) = f(x)$ for all x). Thus, all information about f is determined by its values on $[-L,L]$. We can then define a Fourier series for f and the previous results are immediately adapted to this case. In fact, any function on $(-L,L)$ can be extended to a periodic function:





Fourier series are often used to study periodic functions. They also appear in several applications (e.g., signal processing)

Relation between series on $[-L, L]$ and $[0, L]$

As we saw, in separation of variables, series of the form $\sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$ and $\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ need to be considered on $[0, L]$. These series are related to Fourier series on $[-L, L]$ as follows. Consider first the case

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$ on $[0, L]$. We extend f to an even function on $[-L, L]$ by $\tilde{f}(x) = \begin{cases} f(x), & 0 \le x \le L \\ f(-x), & -L \le x < 0 \end{cases}$. We can now compute the Fourier series of $\tilde{f}$ on $[-L, L]$. But because $\tilde{f}$ is an even function, we have $a_n = \frac{1}{L}\int_{-L}^{L} \tilde{f}(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$

and $b_n = 0$, which is what we had for f on $[0, L]$. Since $\tilde{f}(x) = f(x)$ for $0 \le x \le L$, the Fourier series defined on $[-L, L]$ agrees with the one we had defined on $[0, L]$ for each $x \in [0, L]$.

Similarly, if $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ on $[0,L]$, we can extend f to an odd function on $[-L,L]$ by $\tilde{f}(x) = \begin{cases} f(x), & 0 \le x \le L \\ -f(-x), & -L \le x < 0 \end{cases}$. Computing the Fourier series on $[-L,L]$ and using that $\tilde{f}$ is an odd function, we find

$a_n = 0$ and $b_n = \frac{1}{L}\int_{-L}^{L} \tilde{f}(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$,

so again the results on $[-L,L]$ and on $[0,L]$ agree for $x \in [0,L]$.

Convergence of solutions to the wave equation

We haven't yet showed that the formal solution

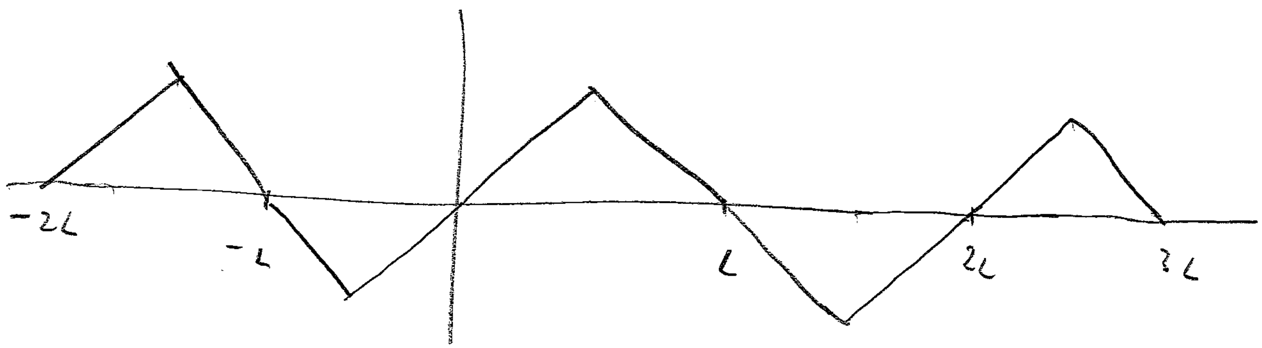
$$u(x,t) = \sum_{n \geq 1}^{\infty} \left(a_n \cos\left(\frac{cn\pi t}{L}\right) + b_n \sin\left(\frac{cn\pi t}{L}\right) \right) \sin\left(\frac{n\pi x}{L}\right)$$

where $a_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$, $b_n = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$ $\left(\begin{array}{l} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{array} \right)$

indeed provides a solution (classical or generalized) to the problem. There are different ways to do this, depending on the properties of f and g. But it should come as no surprise that we need to use results about convergence of Fourier series.

Let us assume that f and g are piecewise C^1 functions and continuous functions.

Recall that f and g are defined on $[0,L]$ and that $f(0)=g(0)=0$. (because of compatibility conditions). We can thus make an odd extension of f and g to odd $2L$-periodic functions defined on $\mathbb{R}$. Furthermore, since $f(L)=g(L)=0$ (again by compatibility conditions) we have $f(-L)=g(-L)=0$



Denote by $\tilde{f}$ and $\tilde{g}$ the odd extensions of f and g, respectively.

Consider the problem

$$\begin{cases} \tilde{u}_{tt} - c^2 \tilde{u}_{xx} = 0 & \text{in } (-\infty,\infty)\times(0,\infty) \\ \tilde{u}(x,0) = \tilde{f}(x) \\ \tilde{u}_t(x,0) = \tilde{g}(x) \end{cases}$$

The solution to this problem is given by D'Alembert's formula:

$$\tilde{u}(x,t) = \frac{\tilde{f}(x+ct) + \tilde{f}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{g}(y)\,dy$$

Using our convergence result for Fourier series (adapted to periodic functions, as discussed) we have

$$\tilde{f}(y) = \frac{\tilde{a}_0}{2} + \sum_{n=1}^{\infty}\left(\tilde{a}_n \cos\left(\frac{n\pi y}{L}\right) + \tilde{b}_n \sin\left(\frac{n\pi y}{L}\right)\right)$$

But since $\tilde{f}$ is odd: $\tilde{a}_n = 0$, $\tilde{b}_n = \frac{1}{L}\int_{-L}^{L} \tilde{f}(y)\sin\left(\frac{n\pi y}{L}\right)dy = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi y}{L}\right)dy$ (*)

Thus:

$$\tilde{f}(y) = \sum_{n=1}^{\infty} \tilde{b}_n \sin\left(\frac{n\pi y}{L}\right)$$

with $\tilde{b}_n$ given by (*). We can thus compute

$$\frac{\tilde{f}(x+ct) + \tilde{f}(x-ct)}{2} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\tilde{b}_n \sin\left(\frac{n\pi(x+ct)}{L}\right) + b_n \sin\left(\frac{n\pi(x-ct)}{L}\right)\right)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} \tilde{b}_n \left(\sin\frac{n\pi x}{L}\cos\frac{n\pi c t}{L} + \cancel{\sin\frac{n\pi ct}{L}\cos\frac{n\pi x}{L}} + \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} - \cancel{\sin\frac{n\pi ct}{L}\cos\frac{n\pi x}{L}} \right)$$

$$= \sum_{n=1}^{\infty} \tilde{b}_n \cos\frac{n\pi ct}{L}\sin\frac{n\pi x}{L},$$ where $\tilde{b}_n$ is given by (*)

Similarly, we have $\tilde{g}(y) = \frac{\tilde{A}_0}{2} + \sum_{n=1}^{\infty}\left(\tilde{A}_n\cos\left(\frac{n\pi y}{L}\right) + \tilde{B}_n \sin\left(\frac{n\pi y}{L}\right)\right)$. Again, since $\tilde{g}$ is odd: $\tilde{A}_n = 0$, $\tilde{B}_n = \frac{1}{L}\int_{-L}^{L} g(y)\sin\left(\frac{n\pi y}{L}\right)dy = \frac{2}{L}\int_0^L g(y)\sin\left(\frac{n\pi y}{L}\right)dy$ (**)

Under our assumptions on g it is valid to integrate the Fourier series term-by-term:

$$\frac{1}{2c}\int_{x-ct}^{x+ct}\sum_{n=1}^{\infty}\tilde{B}_n \sin\left(\frac{n\pi y}{L}\right)dy = \frac{1}{2c}\sum_{n=1}^{\infty}\tilde{B}_n\int_{x-ct}^{x+ct}\sin\left(\frac{n\pi y}{L}\right)dy = \frac{1}{2c}\sum_{n=1}^{\infty} -\tilde{B}_n \frac{L}{n\pi}\cos\left(\frac{n\pi y}{L}\right)\Bigg|_{x-ct}^{x+ct}$$

$$= -\frac{1}{2c}\sum_{n=1}^{\infty}\frac{\tilde{B}_n L}{n\pi}\left(\cos\left(\frac{n\pi(x+ct)}{L}\right) - \cos\left(\frac{n\pi(x-ct)}{L}\right)\right) = -\frac{1}{2c}\sum_{n=1}^{\infty}\frac{\tilde{B}_n L}{n\pi}\left(\cancel{\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)}\right.$$

$$\left. - \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right) - \cancel{\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)} - \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)\right)$$

$$= \sum_{n=1}^{\infty} \frac{\tilde{B}_n L}{n\pi c} \sin\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right).$$ Hence:

$$\tilde{u}(x,t) = \sum_{n=1}^{\infty} \underbrace{\tilde{b}_n}_{\text{call this } a_n} \cos\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} \underbrace{\frac{B_n L}{n\pi c}}_{\text{call this } b_n} \sin\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right) \right) \sin\left(\frac{n\pi x}{L}\right)$$

where from (*) and (**) we have $a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$, $b_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$.

But this is exactly the series solution $u(x,t)$, so $\tilde{u}(x,t) = u(x,t)$ for $0 \le x \le L$ and we conclude that the series for u converges since that for $\tilde{u}$ does. Furthermore, $\tilde{u}$ is odd and satisfies $\tilde{u}(0,t) = \tilde{u}(L,t) = 0$, so $\tilde{u}$ and thus u satisfies the boundary conditions. We conclude that the formal solution u is an actual solution.

<u>Remark</u>: in practice, for "nice" initial data, we expect formal solutions to be actual solutions.