By linearity, any finite sum $\sum_{n=1}^{N}\left(a_n \cos\left(\frac{cn\pi}{L}t\right) + b_n \sin\left(\frac{cn\pi t}{L}\right)\right)\sin\left(\frac{n\pi}{L}x\right)$ is also a solution of the wave equation and it satisfies the boundary conditions. Since this is true for any $N \geq 1$, we are led to consider the series:

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{cn\pi}{L}t\right) + b_n \sin\left(\frac{cn\pi t}{L}\right)\right)\sin\left(\frac{n\pi}{L}x\right)$$

This will only be meaningful if the series converges. Let's assume for now not only that the series converges, but also that we can treat it as a finite sum, in the sense that we can differentiate / integrate term by term, change the summation order, etc. We will discuss these points later on.

Since $u(x,0) = f(x)$ we have

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

Differentiating u w.r.t. t:

$$u_t(x,t) = \sum_{n=1}^{\infty} \left(-a_n \frac{cn\pi}{L} \sin\left(\frac{cn\pi}{L}t\right) + b_n \frac{cn\pi}{L} \cos\left(\frac{cn\pi}{L}t\right)\right) \sin\left(\frac{n\pi}{L}x\right)$$

Since $u_t(x,0) = g(x)$:

$$\sum_{n=1}^{\infty} b_n \frac{cn\pi}{L} \sin\left(\frac{n\pi}{L}x\right) = g(x).$$

Recall that f and g are given. The above expressions tell us that f and g can be written as a sum (series) of trigonometric functions, and we want to use this information to find a_n and b_n. For this, it is useful to first understand the general question of how to write a given function as a sum (series) of trigonometric functions. This is the theme of <u>Fourier series</u>.

Fourier series

Let h be a function defined on $[0,L]$. Suppose that we want to write h as

$$h(x) = \sum_{n=0}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L} x\right) + b_n \sin\left(\frac{n\pi}{L} x\right)\right), \quad (*)$$

(In the expressions for f and g we had no cos, but it is useful to consider this case here.)

where a_n and b_n are real numbers.

We assume that the series converges and can be treated (differentiated, integrated, rearranged, etc) as a finite sum, returning to this point later.

Equality $(*)$ will hold provided we can find coefficients a_n and b_n that make the series equal to h. In order to motivate what follows, suppose that we want to find coefficients a_1, a_2, a_3 such that a vector v can be written as

$$v = a_1 e_1 + a_2 e_2 + a_3 e_3,$$

where e_1, e_2, e_3 are the standard canonical vectors $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$.

We can easily compute a_1 using the dot product:

$$\underbrace{e_1 \cdot v}_{\text{known}} = a_1 \underbrace{e_1 \cdot e_1}_{=1} + a_2 \underbrace{e_1 \cdot e_2}_{=0} + a_3 \underbrace{e_1 \cdot e_3}_{=0} = a_1 \Rightarrow a_1 = e_1 \cdot v$$

Similarly, $a_2 = e_2 \cdot v$, $a_3 = e_3 \cdot v$. Thus

$$v = (e_1 \cdot v) e_1 + (e_2 \cdot v) e_2 + (e_3 \cdot v) e_3 .$$

If we are in N-dimensions and $e_i = (0, 0, \dots, \underset{\substack{\uparrow \\ j^{th} \text{ entry}}}{1}, 0, \dots, 0)$ and we want $v = a_1 e_1 + \dots + a_N e_N$, the same argument yields $a_i = e_i \cdot v$.

Furthermore, suppose that $\{f_n\}_{n=1}^{N}$ is an orthogonal basis, i.e., $f_n \cdot f_m \neq 0$, $f_n \cdot f_m = 0$ whenever $n \neq m$ (we don't assume $f_n \cdot f_n = 1$), and we want to write $v = a_1 f_1 + \dots + a_N f_N = \sum_{n=1}^{N} a_n f_n$. Taking the dot product with f_m:

$$f_m \cdot v = \sum_{n=1}^{N} a_n \underbrace{f_m \cdot f_n}_{=0 \text{ except when } n=m} = a_m f_m \cdot f_m , \text{ thus } a_m = \frac{f_m \cdot v}{f_m \cdot f_m} \text{ for each } m = 1, \dots, N.$$

Suppose now that we have $N = \infty$, ie, we are in "infinite dimensions". It is not difficult to make this idea mathematically precise, but we won't need it here (at this stage we are simply motivating how we want to find the coefficients a_n and b_n). Just assume that the series:

$$v = \sum_{n=1}^{\infty} a_n f_n$$ makes sense and that the vectors f_n satisfy $f_n \cdot f_m = 0$ if $n \neq m$ and $f_m \cdot f_m \neq 0$. Then, exactly as in the case N finite, we compute the coefficients a_n by $a_n = \dfrac{f_n \cdot v}{f_n \cdot f_n}$, so

$$v = \sum_{n=1}^{\infty} \left(\frac{f_n \cdot v}{f_n \cdot f_n} \right) f_n$$

Let's go back to (*), and suppose first that $b_n = 0$:

$h(x) = \sum_{n=0}^{\infty} a_n \cos(\frac{n\pi}{L} x)$. Think of the functions $\cos(\frac{n\pi}{L} x)$ as the "vectors" f_n, so set $f_n(x) = \cos(\frac{n\pi}{L} x)$ and write

$$h = \sum_{n=0}^{\infty} a_n f_n$$

Exactly as in the example with vectors, we can compute a_n if we can find a "product" such that $f_n \cdot f_m = 0$ for $n \neq m$ and $f_n \cdot f_n \neq 0$. In this case $a_n = \frac{f_n \cdot h}{f_n \cdot f_n}$. The difference here is that, because h and the f_n's are functions, our product will not be ordinary multiplication but a more general operation which will consist of <u>multiplication followed by integration</u>.

<u>Def.</u> If f and g are two functions defined on $[a,b]$, we define their <u>inner product</u>, denoted $\langle f,g \rangle$, by

$$\langle f,g \rangle = \int_a^b f(x) g(x)\, dx,$$ provided that the integral makes sense.

Recall that the dot product between vectors is also called an inner product, and the choice of terminology for $\langle f,g \rangle$ is not a coincidence. In fact, $\langle f,g \rangle$ shares many properties with the dot product:

1) $\langle f,g \rangle$ is a real number (not a function, same way that $v \cdot u$ is not a vector)

2) $\langle f,g \rangle = \langle g,f \rangle$

3) $\langle f, cg + dh \rangle = c\langle f,g \rangle + d\langle f,h \rangle$, a,b constant,

4) $\langle cf + dg, h \rangle = c\langle f,h \rangle + d\langle g,h \rangle$, a,b constants.

5) $\langle f, 0 \rangle = 0$, where 0 in $\langle f, 0 \rangle$ is the zero function.

6) $\langle f,f \rangle \geq 0$

7) $\langle f,f \rangle = 0$ if and only if $f = 0$. This property is not entirely true, as it can be seen by taking $f(x) = \begin{cases} 1, & \text{if } x = 1/2 \\ 0, & \text{otherwise,} \end{cases}$; $[a,b] = [0,1]$. But the property is true for all "nice" (e.g. continuous) functions we will be interested in.

Going back to $h = \sum_{n=0}^{\infty} a_n f_n$, $f_n(x) = \cos\left(\frac{n\pi}{L} x\right)$. A direct computation gives

$$\langle f_n, f_m \rangle = \int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right) dx = \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n \neq 0 \\ L & m = n = 0 \end{cases}$$

Therefore, multiplying both sides of $h = \sum_{n=0}^{\infty} a_n f_n$ by f_m and integrating from 0 to L, or, equivalently, taking the inner product of h with f_m yields:

$$\langle h, f_m \rangle = \sum_{n=0}^{\infty} a_n \underbrace{\langle f_m, f_n \rangle}_{0 \; m\neq n, \; \frac{L}{2} \; m=n\neq 0, \; L \; m=n=0} \implies a_n = \frac{\langle h, f_n \rangle}{L/2} \quad n \geq 1, \quad a_0 = \frac{\langle h, f_0 \rangle}{L}$$

or, more explicitly $\boxed{a_n = \frac{2}{L}\int_0^L h(x)\cos\left(\frac{n\pi x}{L}\right) dx, \; n \geq 1,}$ $\boxed{a_0 = \frac{1}{L}\int_0^L h(x)\, dx}$

and

$$h(x) = \frac{1}{L}\int_0^L h(x)\, dx + \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L h(x)\cos\left(\frac{n\pi x}{L}\right) dx\right)\cos\left(\frac{n\pi x}{L}\right)$$

We now go back to (*) and consider $a_n = 0$, so

$$h(x) = \underbrace{\sum_{n=1}^{\infty}}_{} b_n \sin\left(\frac{n\pi x}{L}\right). \quad \text{Put } g_n(x) = \sin\left(\frac{n\pi x}{L}\right) \text{ so } h = \sum_{n=1}^{\infty} b_n g_n$$

we can ignore $n=0$ because $\sin 0 = 0$

A direct computation shows that $\langle g_n, g_m \rangle = \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0, & m \neq n \\ \frac{L}{2}, & m = n \end{cases}$

Hence:

$$\langle h, g_m \rangle = \sum_{n=1}^{\infty} b_n \underbrace{\langle g_n, g_m \rangle}_{0\ m\neq n,\ \frac{L}{2}\ m=n} \Rightarrow b_n = \frac{\langle h, g_n \rangle}{L/2}, \text{ or, more explicitly:}$$

$$\boxed{b_n = \frac{2}{L}\int_0^L h(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 1}$$

and in this case

$$h(x) = \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L h(x)\sin\left(\frac{n\pi x}{L}\right)dx\right)\sin\left(\frac{n\pi x}{L}\right)$$

Conclusion: given a function h defined on $[0,L]$, we learned how to write h as an infinite series of sines or cosines

$$h(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{L} x\right) \quad \text{and} \quad h(x) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi}{L} x\right)$$

where a_n and b_n are given by the above formulas. The series on the right hand sides of these expressions are known as Fourier series for the function h. a_n and b_n are called Fourier coefficients.

Remark: • Note that we haven't discussed (x) when both a_n and b_n are non-zero. We will come back to this point, but the cases discussed so far are enough for solving the PDEs we are interested in.

• We will give a more formal definition of Fourier series later on.

Going back to the wave equation, we have

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \Rightarrow a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$\sum_{n=1}^{\infty} \underbrace{b_n \frac{cn\pi}{L}}_{\tilde{b}_n} \sin\left(\frac{n\pi x}{L}\right) = g(x) \Rightarrow b_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx.$$

Hence, we obtain that the solution to the given initial-boundary value problem for the wave equation is:

$$u(x,t) = \sum_{n=1}^{\infty}\left[\left(\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right)\cos\left(\frac{cn\pi}{L}t\right) + \left(\frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx\right)\sin\left(\frac{cn\pi}{L}t\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

Other boundary conditions

Instead of $u(0,t) = u(L,t) = 0$, we could have chosen $u_x(0,t) = u_x(L,t) = 0$. This boundary condition means that the endpoints of the string move freely.

The B.C. $u(x_0, t) = 0$, x_0 fixed, is called a Dirichlet Boundary Condition, whereas $u_x(x_0,t) = 0$ is called a Neumann Boundary Condition. Note that we can have Dirichlet at one end, and Neumann at another end of the string.