The wave equation in $\mathbb{R}^n$

We will study the initial-value problem:

$$(*) \begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \\ u_t = h & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

Note that we set the speed c equal to 1. $(c^2 = 1)$.

Our goal is to derive a formula for u in terms of g and h, generalizing to $n > 1$ the D'Alembert formula.

Notation For $x \in \mathbb{R}^n$, $t > 0$, and $r > 0$, define

$$U(x,r,t) = \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B_r(x)} u(y,t)\, dS(y) = \frac{1}{\mathrm{vol}(\partial B_r(x))} \int_{\partial B_r(x)} u(y,t)\, dS(y)$$

$$G(x,r) = \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B_r(x)} g(y)\, dS(y) = \frac{1}{\mathrm{vol}(\partial B_r(x))} \int_{\partial B_r(x)} g(y)\, dS(y).$$

$$H(x,r) = \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B_r(x)} h(y)\, dS(y) = \frac{1}{\mathrm{vol}(\partial B_r(x))} \int_{\partial B_r(x)} h(y)\, dS(y)$$

Proposition Suppose that $n \geq 2$, and $u \in C^k(\mathbb{R}^n \times [0,\infty))$ solves (*), where $k \geq 2$. Then $U \in C^k(\mathbb{R}_+ \times [0,\infty))$ and, for fixed $x \in \mathbb{R}^n$,

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ U = G & \text{on } \mathbb{R}_+ \times \{t=0\} \\ U_t = H & \text{on } \mathbb{R}_+ \times \{t=0\} \end{cases}$$

Remark:
- As a function of r, U is defined only on $\mathbb{R}_+$ since $r \geq 0$.
- Notice that U satisfies an equation in $\mathbb{R}_+$ rather than $\mathbb{R}^n$. We are "reducing" the equation to one spatial dimension.
- Recall that $\Delta = \partial_r^2 + \frac{2}{r}\partial_r + \Delta_{S^2}$ in $n=3$. Similarly, for $n \geq 2$,

$\Delta = \partial_r^2 + \frac{n-1}{r}\partial_r + \Delta_{\partial B_1(0)}$. Thus $U_{rr} + \frac{n-1}{r} U_r = \Delta U$ since $\Delta_{\partial B_1(0)} U = 0$.

Thus $U_{tt} - U_{rr} - \frac{n-1}{r} U_r = U_{tt} - \Delta U = 0$, which is the wave equation in polar/spherical coordinates for U

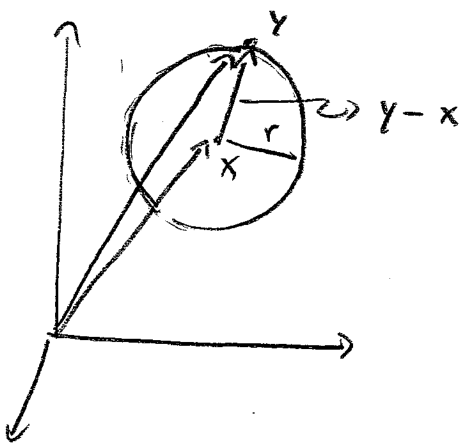
proof. Changing variables $y = x + rz$, we have

$$\mathcal{U}(x,r,t) = \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B_r(x)} u(y,t)\, \underbrace{dS(y)}_{r^{n-1} d\Omega} = \frac{1}{n\alpha(n)} \int_{\partial B_1(0)} u(x+rz,t)\, \overbrace{dS(z)}^{d\Omega(z)}.$$

Assume that we have showed that $\mathcal{U} \in C^k(\mathbb{R}_+ \times [0,\infty))$. Then

$$\frac{\partial}{\partial r} \mathcal{U}(x,r,t) = \frac{1}{n\alpha(n)} \int_{\partial B_1(0)} \frac{\partial}{\partial r}\big(u(x+rz,t)\big)\, dS(z) = \frac{1}{n\alpha(n)} \int_{\partial B_1(0)} \nabla u(x+rz,t) \cdot \frac{\partial}{\partial r}(x+rz)\, dS(z)$$

$$= \frac{1}{n\alpha(n)} \int_{\partial B_1(0)} \nabla u(x+rz,t) \cdot z\, dS(z) \underset{z = \frac{y-x}{r}}{=} \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B_r(x)} \nabla u(y,t) \cdot \underbrace{\left(\frac{y-x}{r}\right)}_{=\nu} dS(y)$$



$$\underset{\substack{\text{Green's}\\ \text{identity}}}{=} \frac{1}{n\alpha(n) r^{n-1}} \int_{B_r(x)} \Delta u(y)\, dy = \frac{r}{n} \frac{1}{\alpha(n) r^n} \int_{B_r(x)} \Delta u(y)\, dy$$

$$= \frac{r}{n} \frac{1}{\mathrm{vol}(B_r(x))} \int_{B_r(x)} \Delta u(y)\, dy.$$

Thus $\mathcal{U}_r = \frac{r}{n}\frac{1}{vol(B_r(x))}\int_{B_r(x)} u_{tt} = \frac{1}{n\alpha(n)}\frac{1}{r^{n-1}}\int_{B_r(x)} u_{tt}$ or

$r^{n-1}\mathcal{U}_r = \frac{1}{n\alpha(n)}\int_{B_r(x)} u_{tt}$. Taking ∂_r: $\partial_r(r^{n-1}\mathcal{U}_r) = \frac{1}{n\alpha(n)}\partial_r\int_{B_r(x)} u_{tt} = \frac{1}{n\alpha(n)}\int_{\partial B_r(x)} u_{tt}$

But $\partial_r(r^{n-1}\mathcal{U}_r) = r^{n-1}\mathcal{U}_{rr} + (n-1)r^{n-2}\mathcal{U}_r$, so

$$\mathcal{U}_{rr} + \frac{n-1}{r}\mathcal{U}_r = \frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B_r(x)} u_{tt} = \partial_t^2\Big(\overbrace{\frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B_r(x)} u}^{=\mathcal{U}}\Big)$$

It remains to show that $\mathcal{U} \in C^k(\mathbb{R}_+ \times [0,\infty))$. We showed that

$\mathcal{U}_r = \frac{r}{n}\frac{1}{vol(B_r(x))}\int_{B_r(x)} \Delta u(y)\,dy$, which gives $\mathcal{U}_r \to 0$ as $r \to 0$. Taking ∂_r again

$\mathcal{U}_{rr} = \frac{1}{vol(\partial B_r(x))}\int_{\partial B_r(x)} \Delta u + \left(\frac{1}{n}-1\right)\frac{1}{vol(B_r(x))}\int \Delta u \xrightarrow[r\to 0]{} \frac{1}{n}\Delta u$. Taking further derivatives we show that $\mathcal{U}_{rrr}$ etc are defined for $0 \le r < \infty$, finishing the proof. $\boxtimes$

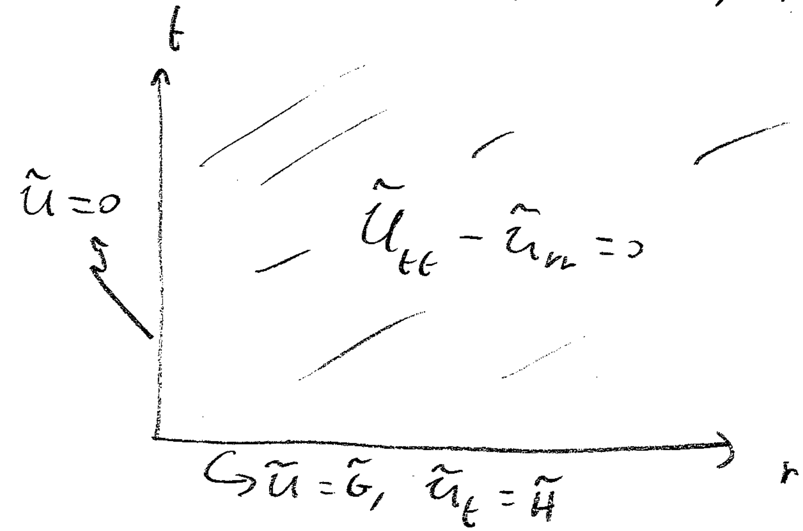
<u>Terminology</u>: the equation $u_{tt} - u_{rr} - \frac{n-1}{r} u_r = 0$ is called the Euler–Poisson–Darboux equation.

We now proceed to solve (*). The method is different depending on whether n is even or odd, and since it is long for arbitrary n, we focus first on $n=3$, which is also the case of primary importance in physics. Although we will keep n as general in the following formulas for future reference.

If $u \in C^2(\mathbb{R}^n \times [0,\infty))$ solves (*), we set $\tilde{u} = r u$, $\tilde{G} = rG$, $\tilde{H} = rH$.

Then $\tilde{u}$ solves (using $n=3$)

$$\begin{cases} \tilde{u}_{tt} - \tilde{u}_{rr} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ \tilde{u} = \tilde{G} & \text{on } \mathbb{R}_+ \times \{t=0\} \\ \tilde{u}_t = \tilde{H} & \text{on } \mathbb{R}_+ \times \{t=0\} \\ \tilde{u} = 0 & \text{on } \{r=0\} \times (0,\infty) \end{cases}$$



This is a 1d wave equation on the half-line with zero boundary condition.

This problem was solved in a HW using a "reflection" through $r=0$. The solution is:

$$\tilde{U}(x,r,t) = \begin{cases} \dfrac{\tilde{G}(r+t) + \tilde{G}(r-t)}{2} + \dfrac{1}{2}\displaystyle\int_{r-t}^{r+t} \tilde{H}(y)\,dy & \text{for } r \geq t \geq 0 \\ \dfrac{\tilde{G}(r+t) - \tilde{G}(t-r)}{2} + \dfrac{1}{2}\displaystyle\int_{-r+t}^{r+t} \tilde{H}(y)\,dy & \text{for } 0 \leq r \leq t \end{cases}$$

where we wrote $\tilde{G}(z)$ and $\tilde{H}(z)$ for $\tilde{G}(x,z)$, $\tilde{H}(x,z)$. But:

$$\lim_{r\to 0} U(x,r,t) = \lim_{r\to 0} \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B_r(x)} u(y,t)\,dy = u(x), \text{ thus,}$$

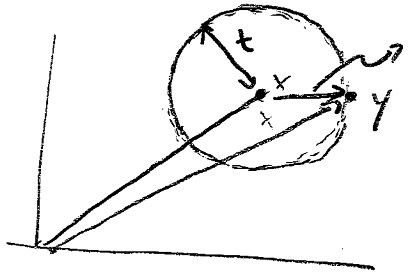
$$u(x) = \lim_{r\to 0} U(x,r,t) = \lim_{r\to 0} \frac{\tilde{U}(x,r,t)}{r} = \lim_{r\to 0}\left[\frac{\tilde{G}(t+r) - \tilde{G}(t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{t+r} \tilde{H}(y)\,dy \right]$$

$$= \tilde{G}'(t) + \tilde{H}(t)$$

Since $G(x,r) = \dfrac{1}{n\alpha(n) r^{n-1}} \displaystyle\int_{\partial B_r(x)} g(y)\,dS(y)$, we have $G(t) = G(x,t) = \dfrac{1}{n\alpha(n) t^{n-1}} \displaystyle\int_{\partial B_t(x)} g(y)\,dS(y)$

so $\tilde{G}'(t) = \dfrac{\partial}{\partial t}(t\,G(t)) = \dfrac{\partial}{\partial t}\left(\dfrac{t}{\mathrm{vol}(B_t(x))}\displaystyle\int_{\partial B_t(x)} g(y)\,dS(y)\right)$ and $\tilde{H}(t) = \dfrac{t}{\mathrm{vol}(B_t(x))}\displaystyle\int_{\partial B_t(x)} h(y)\,dS(y)$.

Changing variables, $\displaystyle \frac{1}{\alpha(n) n t^{n-1}} \int_{\partial B_t(x)} g(y)\, \overbrace{dS(y)}^{t^{n-1}\, d\Omega} \underset{y = x+tz}{=} \frac{1}{\alpha(n) n} \int_{\partial B_1(0)} g(x+tz)\, dS(z)$, so



$$\frac{\partial}{\partial t}\left(\frac{1}{\alpha(n) n t^{n-1}} \int_{\partial B_t(x)} g(y)\, dS(y) \right) = \frac{1}{\alpha(n) n} \int_{\partial B_1(0)} \overbrace{\frac{\partial}{\partial t}\big(g(x+tz) \big)}^{= \nabla g(x+tz) \cdot \frac{\partial}{\partial t}(x+tz)} dS(z)$$

$$= \frac{1}{\alpha(n) n} \int_{\partial B_1(0)} \nabla g(x+tz) \cdot z \, dS(z) \underset{x+tz=y}{=} \frac{1}{n \alpha(n) t^{n-1}} \int_{\partial B_t(x)} \nabla g(y) \cdot \left(\frac{y-x}{t} \right) dS(y)$$

Therefore, we obtain the following formula for the solution of (*) in $n=3$:

$$u(x) = \frac{t}{n \alpha(n) t^{n-1}} \int_{\partial B_t(x)} h(y)\, dS(y) + \frac{1}{n \alpha(n) t^{n-1}} \int_{\partial B_t(x)} g(y)\, dS(y) + \frac{1}{n \alpha(n) t^{n-1}} \int_{\partial B_t(x)} \nabla g(y) \cdot (y-x)\, dS(y)$$

This formula is known as Kirchhoff's formula, and is a generalization of D'Alembert's formula to $n=3$.

For $n=2$, the trick is to use the solution to the three-dimensional problem. We do this by defining $\bar{u} : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ by

$$\bar{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t) \quad \text{where } u \text{ is the sought for solution in 2d.}$$

We also put $\bar{g}(x_1, x_2, x_3) = g(x_1, x_2)$ and $\bar{h}(x_1, x_2, x_3) = h(x_1, x_2)$. Then, since $\bar{u}$ does not depend on x_3, if u solves the wave equation in 2d, $\bar{u}$ solves it in 3d. We can then use Kirchhoff's formula to write an expression for $\bar{u}$, and then obtain u. We will find that

$$u(x,t) = \frac{1}{2} \frac{1}{\alpha(n) t^n} \int_{B_t(x)} \frac{t g(y) + t^2 h(y) + t \nabla g(y) \cdot (y-x)}{\sqrt{t^2 - |y-x|^2}} \, dy$$

is the solution to $(*)$ in $n=2$. We will leave the proof of this fact as an exercise.

The strategy for solving the wave equation in dimensions $n > 3$ is the following. We generalize the argument used in $n=3$ for any n odd. Then, given n odd, we find a solution in $n-1$ ($n-1$ = even) dimensions using the solution in n dimensions, similarly to what we did to get the $n=2$ solution out of the $n=3$ solution.

The main point to obtain solutions for $n > 3$, n odd, is to obtain the equation $\tilde{u}_{tt} - \tilde{u}_{rr} = 0$ on the half-line, which we saw is solved by a reflection argument. Now, for $n > 3$, n odd, in order to get $\tilde{u}_{tt} - \tilde{u}_{rr} = 0$ we cannot define $\tilde{u}$ by $\tilde{u} = rU$, as in $n=3$. Instead, we have to use $\tilde{u} = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}U\right)$, $\tilde{G} = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}G\right)$, $\tilde{H} = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}H\right)$, where $n = 2k+1$; $k \geq 1$ (so n is odd). Of course, this agrees with $\tilde{u} = rU$ for $n=3$ ($k=1$).

We then find that, for $n \geq 3$, n odd

$$u(x,t) = \frac{1}{\gamma_n}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-1}\frac{1}{\mathrm{vol}(\partial B_t(x))}\int_{\partial B_t(x)} g(y)\,dS(y)\right)\right.$$

$$\left.+\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\frac{1}{\mathrm{vol}(\partial B_t(x))}\int_{\partial B_t(x)} h(y)\,dS(y)\right)\right]$$

solves the initial value problem for the wave equation. For n even, we find

$$u(x,t) = \frac{1}{\delta_n}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^{n}\frac{1}{\mathrm{vol}(B_t(x))}\int_{B_t(x)}\frac{g(y)}{\sqrt{t^2-|y-x|^2}}\,dy\right)\right.$$

$$\left.+\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^{n}\frac{1}{\mathrm{vol}(B_t(x))}\int_{B_t(x)}\frac{h(y)}{\sqrt{t^2-|y-x|^2}}\,dy\right)\right]$$

where $\gamma_n = 1 \cdot 3 \cdot 5 \cdots (n-2)$ and $\delta_n = 2 \cdot 4 \cdots (n-2) \cdot n$