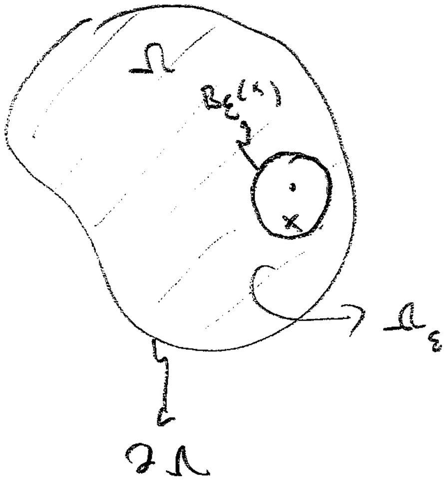
Green's function

Let $\Omega \subseteq \mathbb{R}^n$ be a domain whose boundary $\partial\Omega$ is a regular surface. We want to study the problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

where f and g are given functions. We are going to derive a formula for u in terms of f and g, similar to the formula we found for Poisson's equation.

Let $u \in C^2(\bar{\Omega})$, $x \in \Omega$, and choose $\varepsilon > 0$ small enough such that $B_\varepsilon(x) \subseteq \Omega$. Apply Green's identity in the region $\Omega_\varepsilon = \Omega \setminus B_\varepsilon(x)$



$$\int_{\Omega_\varepsilon} \left(u(y)\, \Delta_y \Gamma(y-x) - \Gamma(y-x)\, \Delta u(y)\right) = \int_{\partial\Omega_\varepsilon} \left(u(y) \frac{\partial \Gamma}{\partial \nu_y}(y-x) - \Gamma(y-x) \frac{\partial u}{\partial \nu}(y)\right) ds(y)$$

It is legitimate to apply Green's identity here because $x \notin \Omega_\varepsilon$, so the derivatives of $\Gamma(y-x)$ are well defined in Ω_ε.

In Ω_ε we know that $\Delta_y \Gamma(y-x) = 0$ (because $\Delta \Gamma(z) = 0$ for $z \neq 0$). Furthermore, $\partial \Omega_\varepsilon = \partial B_\varepsilon(0) \cup \partial \Omega$, and arguing as in the proof of the existence theorem for Poisson's equation, we can show that

$$\lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(x)} \Gamma(y-x) \frac{\partial u}{\partial \nu}(y)\, dS(y) \longrightarrow 0 \quad \text{as } \varepsilon \to 0 \quad \text{and}$$

$$\lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(x)} u(y) \frac{\partial \Gamma(y-x)}{\partial \nu_y}\, dS(y) \longrightarrow u(x) \quad \text{as } \varepsilon \to 0.$$

Therefore, taking $\varepsilon \to 0$ in Green's identity above:

$$-\int_\Omega \Gamma(y-x)\, \Delta u(y)\, dy = u(x) + \int_{\partial\Omega} u(y) \frac{\partial \Gamma(y-x)}{\partial \nu_y}\, dS(y) - \int_{\partial \Omega} \Gamma(y-x) \frac{\partial u}{\partial \nu}(y)\, dS(y)$$

or yet

$$u(x) = \int_{\partial\Omega} \left(\Gamma(y-x) \frac{\partial u}{\partial \nu}(y) - u(y) \frac{\partial \Gamma}{\partial \nu_y}(y-x) \right) dS(y) - \int_\Omega \Gamma(y-x)\, \Delta u(y)\, dy$$

Since we want $u = g$ on $\partial\Omega$ and $-\Delta u = f$ in Ω, we could substitute these equalities in the above formula:

$$u(x) = \int_{\partial\Omega} \left(\Gamma(y-x) \frac{\partial u}{\partial \nu}(y) - g(y) \frac{\partial \Gamma}{\partial \nu_y}(y-x) \right) dS(y) + \int_{\Omega} \Gamma(y-x) f(y)\, dy.$$

This does not quite give a formula for u though: f, g and Γ are known, but the right-hand side also involves $\frac{\partial u}{\partial \nu}$, which we don't know because we are trying to find u.

To solve this problem, we consider, for each fixed $x \in \Omega$, the problem

$$\begin{cases} \Delta_y \varphi(x,y) = 0 & \text{in } \Omega \\ \varphi(x,y) = \Gamma(y-x) & \text{on } \partial\Omega \end{cases}$$

Notice that since $x \in \Omega$, we always have $y - x \neq 0$ for $y \in \partial\Omega$. Thus, $\Gamma(y-x)$ on $\partial\Omega$ is always well defined. Green's identity now gives

$$-\int_{\Omega} \varphi(x,y)\, \Delta u(y)\, dy = \int_{\partial\Omega} \left(u(y) \frac{\partial \varphi}{\partial \nu_y}(x,y) - \varphi(x,y) \frac{\partial u}{\partial \nu}(y) \right) dS(y)$$

$$= \int_{\partial\Omega} \left(u(y) \frac{\partial \varphi}{\partial \nu_y}(x,y) - \Gamma(x-y) \frac{\partial u}{\partial \nu}(y) \right) dS(y).$$

Adding the above formula to the previous formula for u, the terms $\int_{\partial\Omega} \Gamma(x-y) \frac{\partial u}{\partial \nu}(y)\, dS(y)$ cancel out and we obtain

$$u(x) = -\int_{\Omega} G(x,y)\, \Delta u(y)\, dy - \int_{\partial\Omega} u(y) \frac{\partial G}{\partial \nu_y}(x,y)\, dS(y), \quad \text{where}$$

$$G(x,y) = \Gamma(x-y) - \varphi(x,y) \quad (x \neq y).$$

So, if we we want to solve $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$ then we will set:

$$u(x) = -\int_{\Omega} G(x,y) f(y)\, dy - \int_{\partial\Omega} g(y) \frac{\partial G}{\partial \nu_y}(x,y)\, dS(y).$$

__Def.__ G as above is called the Green function (for the domain Ω).

G should be thought of as the analogue of $\Gamma(x-y)$ for the case of a domain Ω with boundary. In fact, by construction G solves:

$$\begin{cases} -\Delta_y G(x,y) = 0 & \text{in } \Omega\setminus\{x\} \quad (\text{i.e., for } y \neq x,\ x,y \in \Omega) \\ G = 0 & \text{on } \partial\Omega \end{cases}$$

<u>Proposition</u> (symmetry of the Green function). For all $x, y \in \Omega$, $x \neq y$, we have $G(x,y) = G(y,x)$.

<u>proof</u>: For $x \neq y$, define $v(z) = G(x,z)$, $w(z) = G(y,z)$, $z \in \Omega$. Then $\Delta v(z) = 0$ $(z \neq x)$, $\Delta w(z) = 0$ $(z \neq y)$ and $v = w = 0$ on $\partial\Omega$. From Green's identity we have

$$\underbrace{\int_{\Omega\setminus(B_\varepsilon(x)\cup B_\varepsilon(y))} (w\Delta v - v\Delta w)}_{=0} = \int_{\partial(\Omega\setminus(B_\varepsilon(x)\cup B_\varepsilon(y)))} \left(w\frac{\partial v}{\partial \nu} - v\frac{\partial w}{\partial \nu}\right) = \underbrace{\int_{\partial\Omega}\left(w\frac{\partial v}{\partial \nu} - v\frac{\partial w}{\partial \nu}\right)}_{=0} + \int_{\partial B_\varepsilon(x)}\left(w\frac{\partial v}{\partial \nu} - v\frac{\partial w}{\partial \nu}\right) + \int_{\partial B_\varepsilon(y)}\left(w\frac{\partial v}{\partial \nu} - v\frac{\partial w}{\partial \nu}\right)$$

Thus $\displaystyle\int_{\partial B_\varepsilon(x)} \left(w \frac{\partial v}{\partial \nu} - v \frac{\partial w}{\partial \nu} \right) = \int_{\partial B_\varepsilon(y)} \left(v \frac{\partial w}{\partial \nu} - w \frac{\partial v}{\partial \nu} \right)$. Arguing as in the proof of the existence theorem for Poisson's equation in $\mathbb{R}^n$, we have

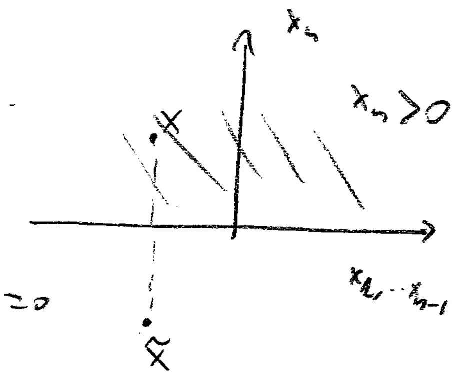
$$\int_{\partial B_\varepsilon(x)} v \frac{\partial w}{\partial \nu} \xrightarrow[\varepsilon \to 0]{} 0, \quad \int_{\partial B_\varepsilon(y)} w \frac{\partial v}{\partial \nu} \xrightarrow[\varepsilon \to 0]{} 0, \quad \int_{\partial B_\varepsilon(x)} w \frac{\partial v}{\partial \nu} \xrightarrow[\varepsilon \to 0]{} w(x), \quad \int_{\partial B_\varepsilon(y)} v \frac{\partial w}{\partial \nu} \xrightarrow[\varepsilon \to 0]{} v(y)$$

Therefore $w(x) = v(y)$ or $w(x) = G(y,x) = G(x,y) = v(y)$. $\boxed{\diagup\!\!\diagdown}$

From the above calculations we see that to solve $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$, we need to find the Green's function for the domain Ω. How to do that? Note that we haven't proved that such a G always exists. While this can be proven, a proof would be beyond the scope of this course. Instead, we will give explicit formulas for G when Ω is a simple domain.

Green's function for the half space We denote by $\mathbb{R}^n_+$ the set

$\mathbb{R}^n_+ = \{ x \in \mathbb{R}^n \mid x_n > 0 \}$, sometimes called the half-space.



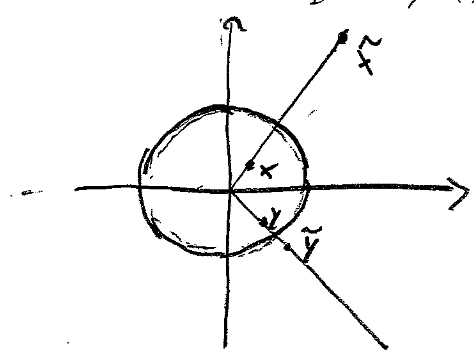
Given $x = (x_1, \dots, x_{n-1}, x_n)$, denote by $\tilde{x}$ the reflection thru $x_n = 0$

$\tilde{x} = (x_1, \dots, x_{n-1}, -x_n)$. The Green function for $\Omega = \mathbb{R}^n_+$ is given by

$$G(x,y) = \Gamma(x-y) - \Gamma(y - \tilde{x}), \quad x, y \in \mathbb{R}^n_+, \quad x \neq y.$$

Green's function for the unit ball Consider $B_1(0) \subseteq \mathbb{R}^n$. Given $x \in B_1(0)$, $x \neq 0$,

let $\tilde{x} = \frac{x}{|x|^2}$, sometimes called the inversion thru $\partial B_1(0)$. The Green function for $\Omega = B_1(0)$ is given by



$$G(x,y) = \Gamma(x-y) - \Gamma(|x|(y - \tilde{x})).$$

Exercise: Show that the above function is fact give the Green function for the domains $\mathbb{R}^n_+$ and $B_1(0)$, respectively.