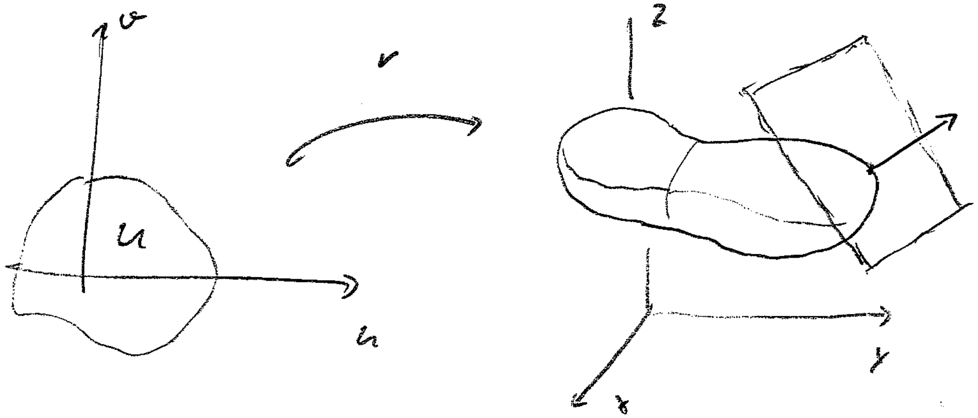
A few tools from calculus in $\mathbb{R}^n$

Recall that in multivariable calculus, a parametric surface $r : U \to \mathbb{R}^3$, $r = r(u,v)$, is called <u>smooth</u> if $r_u \times r_v = \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ is not zero (not the zero vector). This means that the tangent plane to the surface at each point is well-defined, as are normal vectors to the surface.



This concept generalizes to $\mathbb{R}^n$. We say that a surface

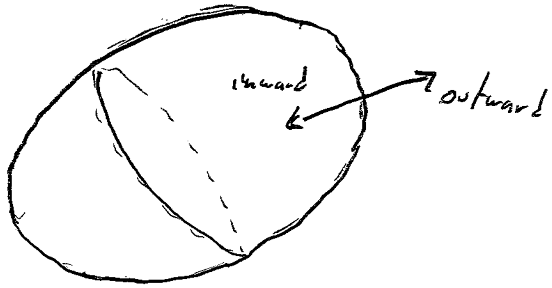
$$r : U \subseteq \mathbb{R}^{n-1} \to \mathbb{R}^n$$

is <u>regular</u> if has a well defined normal vector at each point

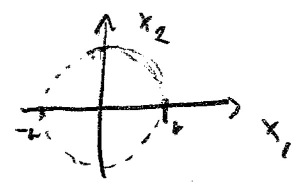
As in 3d calculus, we say that a surface is <u>orientable</u> if it admits unit normal vector field that varies continuously over the surface (recall from calculus that an example of a non-orientable surface is the Möbius strip)

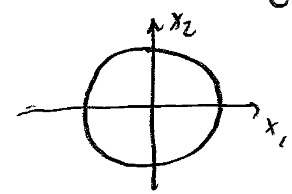
We define an orientable surface in $\mathbb{R}^n$ in exactly the same way. Unless stated otherwise, all surfaces we will dealt with will be assumed orientable.

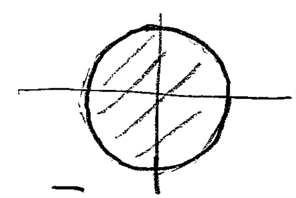
We will be mostly interested in situations where our surface is the boundary $\partial\Omega$ of a <u>domain</u> Ω in $\mathbb{R}^n$, ie, of an open connected set Ω. In this case we can choose normal vectors pointing outwards thus we will assume that normal vectors are outward normal vectors unless stated otherwise



We will denote by $\overline{\Omega}$ (sometimes called the closure of Ω) the set $\overline{\Omega} = \Omega \cup \partial\Omega$







$\Omega = \{x_1^2 + x_2^2 < r^2\}$ $\quad$ $\partial\Omega = \{x_1^2 + x_2^2 = r^2\}$ $\quad$ $\overline{\Omega} = \{x_1^2 + x_2^2 \leq r^2\}$

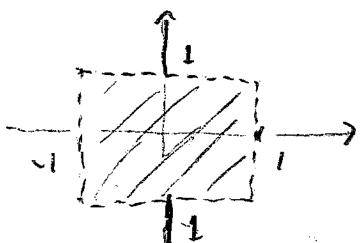
A function f defined on $\overline{\Omega}$, $f: \overline{\Omega} \to \mathbb{R}$, gives rise to a function defined on $\partial\Omega$, called the restriction of f to Ω and denoted by $f|_{\partial\Omega}$, by $f|_{\partial\Omega}(x) = f(x)$, $x \in \partial\Omega$.

EX: $\Omega = \{x_1^2 + x_2^2 < r^2\}$, $f: \Omega \to \mathbb{R}$ $\quad f(x_1, x_2) = x_1$ $\quad f|_{\partial\Omega}(x_1, x_2) = x_1$

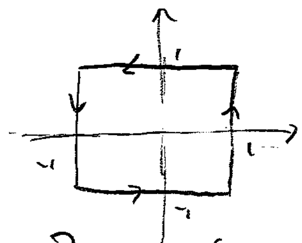
Sometimes we denote $f|_{\partial\Omega}$ simply by f in calculations.

Given $f : \overline{\Omega} \to \mathbb{R}$, we can consider the integral of f over Ω, i.e., $\int_{\overline{\Omega}} f(x)\,dx$, which we sometimes denote simply $\int_{\overline{\Omega}} f$, and also the integral of $f|_{\partial\Omega}$ over, $\int_{\partial\Omega} f(x)\,dS$, where, like in 3d calculus, dS is the volume element on the surface. We sometimes denote $\int_{\partial\Omega} f(x)\,dS$ simply by $\int_{\partial\Omega} f$.

<u>Ex:</u> Ω = square with vertices $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$ and $f(x_1,x_2) = x_2$.



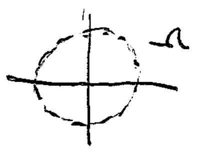
Ω



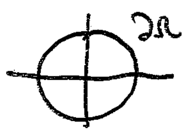
$\partial\Omega$ (orientation as in calculus)

Then
$$\int_{\partial\Omega} f\,dS = \int_{-1}^{1} f(x_1,x_2)\,dx_1 + \int_{-1}^{1} f(x_1,x_2)\,dx_2 + \int_{1}^{-1} f(x_1,x_2)\,dx_1 + \int_{1}^{-1} f(x_1,x_2)\,dx_2 = \int_{-1}^{1} (-1)\,dx_1 + \int_{-1}^{1} x_2\,dx_2 + \int_{1}^{-1} 1\,dx_1 + \int_{1}^{-1} x_2\,dx_2$$
$$= -2 + \tfrac{1}{2}x_2^2\Big|_{-1}^{1} + (-2) + \tfrac{1}{2}x_2^2\Big|_{1}^{-1} = -4$$

<u>Ex:</u> Ω = unit ball, $f(x_1,x_2) = x_1 x_2$. Using polar coordinates $f(r,\theta) = r^2\cos\theta\sin\theta$, $dS = d\theta$,



So
$$\int_{\partial\Omega} f = \int_0^{2\pi} \underbrace{r^2}_{=1} \cos\theta\sin\theta\,d\theta = 0.$$



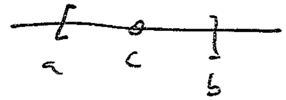
Notation We will denote by $\nu(x)$ outer $\overbrace{\text{unit}}^{\text{length}=1}$ normal vector to $\partial\Omega$ at x. Sometimes we omit the argument x and write simply ν. Note that $\nu = (\nu_1, \dots, \nu_n)$.

Integration by parts formula. Let Ω be a domain in $\mathbb{R}^n$ such that $\partial\Omega$ is a regular surface. Let $f: \bar{\Omega} \to \mathbb{R}$ and $g: \bar{\Omega} \to \mathbb{R}$ be differentiable functions. Then

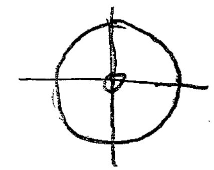
$$\int_\Omega f \frac{\partial g}{\partial x_i} = -\int_\Omega \frac{\partial f}{\partial x_i} g + \int_{\partial\Omega} f g \nu_i$$

Remarks

- The integral over a n-dimensional object remains the same if we remove a $(n-1)$-dimensional set. Eg $\int_a^b f(x)\,dx = \int_{[a,b]\setminus\{c\}} f(x)\,dx$

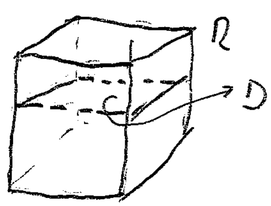


, $\int_B f\,dx = \int_{B\setminus\{0\}} f\,dx$



,

$\int_R f\,dx = \int_{R\setminus D} f\,dx$



Thus, since $\bar{\Omega}$ is n-dimensional and $\partial\Omega$ $(n-1)$-dimensional, we have $\int_{\bar{\Omega}} = \int_{\Omega\cup\partial\Omega} = \int_\Omega$ and the formula can be written: $\int_{\bar{\Omega}} f \frac{\partial g}{\partial x_i} = -\int_{\bar{\Omega}} \frac{\partial f}{\partial x_i} g + \int_{\partial\Omega} f g \nu_i$

- Consider $n=1$, $\Omega=(a,b)$. Then the formula reads:

$$\int_a^b f(x)\,g'(x)\,dx = -\int_a^b f'(x)\,g(x)\,dx + f(b)g(b) - f(a)g(a),$$

which is just the usual integration by parts formula. Note that $f(b)g(b)-f(a)g(a)$ corresponds to $\int_{\partial\Omega} fg\,\nu_i$: $\partial\Omega=\{a\}\cup\{b\}$, $\nu=1$ at $\{b\}$ and $\nu=-1$ at $\{a\}$, and the integral over a point is just the value of the function $\int_{\{b\}} h(x)\,dx = h(b)$.

$$\int_a^b f(x)\,g'(x)\,dx, \quad u=f,\ dv=g\,dx \Rightarrow du=f'\,dx,\ v=g$$

$$\int_a^b f(x)\,g'(x)\,dx = uv\Big|_a^b - \int_a^b v\,du = f(x)g(x)\Big|_a^b - \int_a^b g(x)\,f'(x)\,dx$$

- Recall the divergence theorem in 3d.

$$\iiint_E \operatorname{div}\vec{F}\,dV = \iint_S \vec{F}\cdot d\vec{S}$$

where S is the boundary of E

In our notation, with $E = \Omega$, $S = \partial\Omega$, $dV = dx = dx_1\, dx_2\, dx_3$, it reads,

$$\int_\Omega \operatorname{div} \vec{F}\, dx = \int_{\partial\Omega} \vec{F} \cdot d\vec{S}.$$ But $d\vec{S} = \nu\, dS$ $(= \vec{n}\, dS)$, so, using our notation that omits arrows on vectors: $\int_\Omega \operatorname{div} F\, dx = \int_{\partial\Omega} F \cdot \nu\, dS$

Now put $F = fg(1,0,0) = (fg, 0, 0)$. Then $\operatorname{div} F = \frac{\partial}{\partial x_1}(fg) = \frac{\partial f}{\partial x_1} g + f \frac{\partial g}{\partial x_1}$ and $F \cdot \nu = fg\nu_1$, so $\int_\Omega \left(\frac{\partial f}{\partial x_1} g + g \frac{\partial f}{\partial x_1} \right) dx = \int_{\partial\Omega} fg\nu_1\, dS$, which is the integration by parts formula for $i=1$. With $F = fg(0,1,0)$ and $F = fg(0,0,1)$, we obtain the formula for $i = 2, 3$.

Now, the divergence theorem holds in $\mathbb{R}^n$: $\int_\Omega \operatorname{div} F\, dx = \int_{\partial\Omega} F \cdot \nu\, dS$, $\Omega \subset \mathbb{R}^n$.

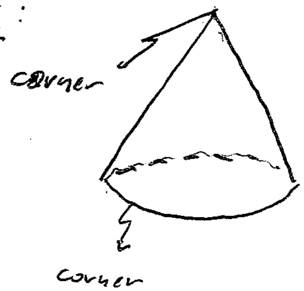
Using $F = fg e_i$, ie, $F = fg(0, 0, \ldots, \underset{\substack{\uparrow \\ i\text{th entry}}}{1}, 0, \ldots, 0)$, and proceeding as above, we obtain the formula in $\mathbb{R}^n$.

• Instead of deriving the integration by parts from the divergence theorem, it is possible to prove the integration by parts formula directly, and then derive not only the divergence theorem but also Green's identities from it.

• We can extend the integration by parts formula to surfaces that are not regular, but such that the normal is defined outside a $n-2$-dimensional subset of $\partial\Omega$ (the "corners" of $\partial\Omega$)

Ex:

cone



Ω = interior of the cone (3d)

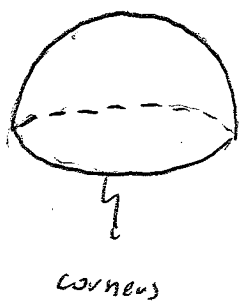
$\partial\Omega$ = surface of the cone (2d)

corners = vertex and the boundary of the base (0d and 1d, respectively)

$$\int_{\partial(\text{cone})} = \int_{\partial(\text{cone}) \setminus \text{corners}}$$

Ex:

half-sphere



Ω = interior of the half-sphere (3d)

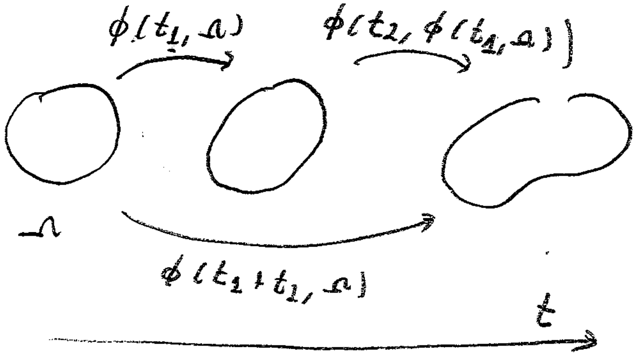
$\partial\Omega$ = surface of the half-sphere (2d)

corners = equator (1d)

$$\int_{\partial(\text{half-sphere})} = \int_{\partial(\text{half-sphere}) \setminus \text{equator}}.$$

Domains evolving in time and involving a parameter

Let $\phi = \phi(t,x) : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ be a differentiable map such that for each fixed t, $\phi(t,\cdot)$ is a one-to-one map that has a differentiable inverse. Given $\Omega \subseteq \mathbb{R}^n$, we think of $\phi(t,\Omega)$ as flowing the domain Ω from $t=0$ to time t. Denote $\phi_t(x) = \phi(t,x)$.



Then

$$\frac{d}{dt}\int_{\phi_t(\Omega)} f(t,x)\,dx = \int_{\phi_t(\Omega)} \frac{\partial f}{\partial t}(t,x)\,dx + \int_{\partial\phi_t(\Omega)} f(t,x)\, v(t,x)\cdot \nu(x)\,dS$$

where v is the velocity of the flow: $\dfrac{\partial \phi(t,x)}{\partial t} = v(t,\phi(t,x))$.

Denoting by $\Omega(t) = \phi_t(\Omega)$, we can write

$$\frac{d}{dt}\int_{\Omega(t)} f = \int_{\Omega(t)} \frac{\partial f}{\partial t} + \int_{\partial\Omega(t)} f\, v\cdot\nu$$

In particular, if f does not depend on time $\frac{d}{dt}\int_{\Omega(t)} f = \int_{\partial\Omega(t)} f\, v\cdot\nu$

As a consequence, if $B_r(x_0)$ is the ball of radius r centered at x_0 in $\mathbb{R}^n$

$$\frac{d}{dr}\int_{B_r(x_0)} f\,dx = \int_{\partial B_r(x_0)} f\,dS$$

Integration in polar coordinates,

$$\int_{\mathbb{R}^n} f\,dx = \int_0^\infty \int_{\partial B_r(x_0)} f\,dS\,dr \quad \text{for each } x_0 \in \mathbb{R}^n$$