Transformation methods

The transformation methods we will study allow us to convert a PDE into either an algebraic equation or else in a PDE with fewer variables. We will study two such tools: the Fourier transform and the Laplace transform.

These methods are often used in physics and engineering where functions can be complex, so it will be convenient to allow for complex-valued functions. It is also convenient to consider functions defined on $\mathbb{R}^n$, i.e., $u = u(x^1, x^2, \ldots, x^n)$

We recall some notation. If $f = f(x) = f(x^1, \ldots, x^n)$, then

$$\int_{\mathbb{R}^n} f(x)\, dx = \int \cdots \int_{\mathbb{R}^n} f(x^1, \ldots, x^n)\, dx^1 \ldots dx^n = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x^1, \ldots, x^n)\, dx^1 \ldots dx^n$$

<u>Remark</u>: In these transformation methods, we will be dealing with integrals over $\mathbb{R}^n$, i.e., with improper integrals of the form $\int_{-\infty}^{+\infty} \ldots$. We will <u>assume</u> that these integrals always converge and that the manipulations we will carry out with them are valid.

Def Let $f : \mathbb{R}^n \to \mathbb{C}$ be such that $\int_{\mathbb{R}^n} |f(x)|\, dx < \infty$. Then we define the Fourier transform of f, denoted $\hat{f}$ or $\mathcal{F}(f)$, as the function: $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ given by

$$\hat{f}(k) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ik\cdot x} f(x)\, dx \,, \quad \text{where } k\cdot x = k_1 x_1 + \dots + k_n x_n \,, \quad i^2 = -1.$$

The inverse Fourier transform of a function $f : \mathbb{R}^n \to \mathbb{C}$, denoted $\check{f}$ or $\mathcal{F}^{-1}(f)$ is defined as:

$$\check{f}(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ik\cdot x} f(k)\, dk.$$

Remark Since $|e^{\pm ik\cdot x}| = 1$, $\left| \int_{\mathbb{R}^n} e^{\pm ik\cdot x} f(x)\, dx \right| \le \int_{\mathbb{R}^n} |f(x)|\, dx < \infty$, thus $\hat{f}$ and $\check{f}$ are well-defined. As said, however, we will omit specific assumptions on the functions and assume convergence of all integrals.

Notation Sometimes we use uppercase letters to denote $\hat{f}$, i.e., $f(x)$ and $F(k) = \hat{f}(k)$.

Ex: Take $n=1$ and $f(x) = e^{-\frac{x^2}{2}}$. Then $\hat{f}(k) = e^{-\frac{k^2}{2}}$

To see this:

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (\cos(kx) + i \sin(kx)) e^{-\frac{x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \cos(kx) e^{-\frac{x^2}{2}} dx + \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sin(kx) e^{-\frac{x^2}{2}} dx$$

These integrals can be computed using more advanced techniques of complex variables and give $\int_{-\infty}^{+\infty} \cos(kx) e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}\, e^{-\frac{k^2}{2}}$, $\int_{-\infty}^{+\infty} \sin(kx) e^{-\frac{x^2}{2}} dx = 0$ (this last integral could be "guessed" from the "symmetric limits", although we need to be careful because these are improper integrals).

Remark: Many of the integrals that appear in transformation methods require more advanced techniques to be computed. From a practical standpoint, one can use softwares like Mathematica or numerical integration to perform such integrals.

Ex: If $f(x) = \begin{cases} 1, & |x| \le a \\ 0, & |x| > 0 \end{cases}$ then $\hat{f}(k) = \sqrt{\frac{2}{\pi}} \frac{\sin(ak)}{k}$

To see this:

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} f(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}} \left. \frac{e^{-ikx}}{-ik} \right|_{-a}^{a} = -\frac{1}{\sqrt{2\pi}} \frac{e^{-ika} - e^{ika}}{ik}$$

Recalling
$$e^{ika} = \cos(ka) + i\sin(ka)$$
$$e^{-ika} = \cos(ka) - i\sin(ka)$$
$\xRightarrow{\text{subtract}}$ $e^{ika} - e^{-ika} = 2i\sin(ka)$

Thus

$$\hat{f}(k) = \frac{2}{\sqrt{2\pi}} \frac{\sin(ka)}{k}$$

Remark: The Fourier transform can be found defined differently in the literature

Eg, without the factor $(2\pi)^{n/2}$, or $\mathcal{F}(f) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{+ik\cdot x} f(x)\, dx$ and

$$\mathcal{F}^{-1}(f) = \frac{1}{(2\pi)^{n/2}} \int e^{-ik\cdot x} \hat{f}(k)\, dk$$

Properties of the Fourier transform

Let a, b be constants, f, g functions, and denote complex conjugation of f by $\bar{f}$, $x = (x_1, \dots, x_n)$, $k = (k_1, \dots, k_n)$

1. $(\hat{f})^{\vee} = f$, or $\mathcal{F}^{-1}(\mathcal{F}(f)) = f$. This is why we call $\mathcal{F}^{-1}$ the inverse Fourier transform.

2. $(af + bg)^{\wedge} = a\hat{f} + b\hat{g}$ or $\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$,

   $(af + bg)^{\vee} = a\check{f} + b\check{g}$ or $\mathcal{F}^{-1}(af + bg) = a\mathcal{F}^{-1}(f) + b\mathcal{F}^{-1}(g)$,

i.e, the Fourier transform is linear. (a linear operator)

3 $\left(\dfrac{\partial f}{\partial x_j}\right)^{\wedge} = i k_j \hat{f}$, $k = (k_1, \dots, k_n)$. More generally

$$\mathcal{F}\left(\frac{\partial^{m_1 + m_2 + \dots + m_n} f}{\partial x_1^{m_1} \partial x_2^{m_2} \dots \partial x_n^{m_n}}\right) = (i k_1)^{m_1} (i k_2)^{m_2} \dots (i k_n)^{m_n} \hat{f}$$

4. $\displaystyle\int_{\mathbb{R}^n} f(x)\, \overline{g(x)}\, dx = \int_{\mathbb{R}^n} \hat{f}(k)\, \overline{\hat{g}(k)}\, dk$

5. $\overline{\hat{f}(k)} = \hat{f}(-k)$ if $f(x)$ is real

6. $\mathcal{F}(f(x-a)) = e^{-ika}\hat{f}(k)$

7. $\mathcal{F}(e^{iax}f(x)) = \hat{f}(k-a)$

8. $\mathcal{F}(f(ax)) = \frac{1}{|a|}\hat{f}\left(\frac{k}{a}\right) \quad (a \neq 0)$

9. $\mathcal{F}(f * g) = (2\pi)^{\frac{n}{2}}\hat{f}\hat{g}$, where $f*g$ is called the convolution of f and g and is the function given by $f*g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,dy$

10. $\mathcal{F}(fg) = \frac{1}{(2\pi)^{\frac{n}{2}}}\hat{f} * \hat{g}$.

In particular, the Fourier transform of a product of two functions is <u>not</u> the product of its Fourier transforms.

Using the Fourier transform to solve PDEs

EX: Solve $-\Delta u + u = f$ in $\mathbb{R}^n$, where f is a given function.

Apply $\mathcal{F}$ on both sides:

$$\mathcal{F}(-\Delta u + u) = -\mathcal{F}(\Delta u) + \mathcal{F}(u) = \mathcal{F}(f)$$ But:

$$\mathcal{F}(\Delta u) = \mathcal{F}\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}\right) = \mathcal{F}\left(\frac{\partial^2 u}{\partial x_1^2}\right) + \cdots + \mathcal{F}\left(\frac{\partial^2 u}{\partial x_n^2}\right)$$

$$= (ik_1)^2 \hat{u} + (ik_2)^2 \hat{u} + \cdots + (ik_n)^2 \hat{u} = -(k_1^2 + \cdots + k_n^2)\hat{u} = -|k|^2 \hat{u}.$$ Thus

$$|k|^2 \hat{u} + \hat{u} = \hat{f} \quad \text{or} \quad \hat{u} = \frac{\hat{f}}{1+|k|^2}$$

Since f is given, $\hat{f}$ is known. Therefore we can find u by computing the inverse Fourier transform:

$$u = \mathcal{F}^{-1}(\hat{u}) = \mathcal{F}^{-1}\left(\frac{\hat{f}}{1+|k|^2}\right)$$

To compute $\mathcal{F}^{-1}\left(\frac{\hat{f}}{1+|k|^2}\right)$, denote by $\hat{g}(k) = \frac{1}{1+|k|^2}$. Then

$$u = \mathcal{F}^{-1}\left(\frac{\hat{f}}{1+|k|^2}\right) = \mathcal{F}^{-1}(\hat{f}\,\hat{g}) = \frac{1}{(2\pi)^{\frac{n}{2}}} f * g, \text{ where } g = \mathcal{F}^{-1}(\hat{g}).$$

Thus, we have an explicit formula for u: $u(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy$,

provided that we can find the function g. For this, note that if $a > 0$,

$$\int_0^\infty e^{-ta}\, dt = \frac{e^{-ta}}{-a}\Big|_0^\infty = \frac{1}{a}.$$ Using this with $a = 1+|k|^2$ (which is >0):

$$\frac{1}{1+|k|^2} = \int_0^\infty e^{-t(1+|k|^2)}\, dt, \text{ i.e., } \hat{g}(k) = \int_0^\infty e^{-t(1+|k|^2)}\, dt$$

$$g(x) = \mathcal{F}^{-1}(\hat{g}(k)) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ik\cdot x}\, \hat{g}(k)\, dk$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ik\cdot x} \int_0^\infty e^{-t(1+|k|^2)}\, dt\, dk$$

Switching the order of integration:

$$g(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_0^\infty e^{-t} \int_{\mathbb{R}^n} e^{ik\cdot x - t|k|^2} \, dk \, dt .$$ The integral $\int_{\mathbb{R}^n} e^{ik\cdot x - t|k|^2} dk$

can be computed using more advanced techniques of complex variables, giving:

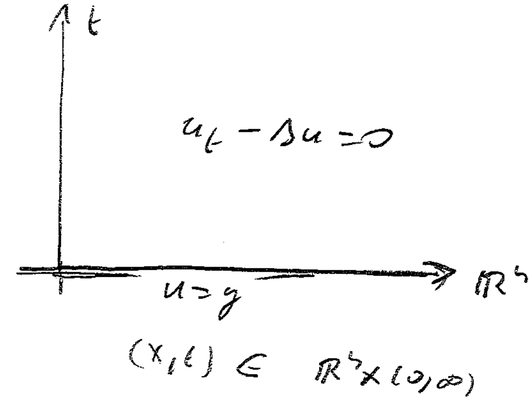
$$\int_{\mathbb{R}^n} e^{ik\cdot x - t|k|^2} \, dk = \left(\frac{\pi}{t}\right)^{n/2} e^{-\frac{|x|^2}{4t}}$$, thus $$g(x) = \frac{1}{2^{n/2}} \int_0^\infty \frac{e^{-t - \frac{|x|^2}{4t}}}{t^{n/2}} \, dt .$$

Using this in the expression for u we find

$$u(x) = \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^\infty \int_{\mathbb{R}^n} \frac{e^{-t - \frac{|x-y|^2}{4t}}}{t^{n/2}} f(y) \, dy \, dt .$$

Ex: Solve the initial value problem for the heat equation

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$



We consider the Fourier transform in $\mathbb{R}^n$, i.e., only for the x-variable. Then, $\mathcal{F}(\Delta u) = -|k|^2 \hat{u}$, $\hat{u} = \hat{u}(t,k)$, and

$$\mathcal{F}(u_t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ik\cdot x} \partial_t u(x)\, dx = \partial_t \left(\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ik\cdot x} u(x)\, dx \right) = \hat{u}_t$$

We also have $\hat{u} = \hat{g}$ on $\mathbb{R}^n \times \{t=0\}$. Therefore $\hat{u}$ solves

$$\begin{cases} \hat{u}_t + |k|^2 \hat{u} = 0 & \text{for } t>0 \\ \hat{u} = \hat{g} \end{cases}$$

which is an ODE for $\hat{u}$. The solution is

$$\hat{u}(k,t) = e^{-t|k|^2} \hat{g}(k)$$

Hence $u(x,t) = \mathcal{F}^{-1}(e^{-t|k|^2}\hat{g}(k)) = \frac{1}{(2\pi)^{n/2}} f(t,x) * g(x)$, where

$f(t,x) = \mathcal{F}^{-1}(e^{-t|k|^2})$ and the convolution involves only the spatial variables $f(t,x) * g(x) = \int_{\mathbb{R}^n} f(t,y)\, g(x-y)\, dy$. Computing:

$$\mathcal{F}^{-1}(e^{-t|k|^2}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ik\cdot x} e^{-t|k|^2}\, dk = \frac{1}{(2t)^{n/2}} e^{-\frac{|x|^2}{4t}}$$

using the result from the previous example.

Thus,

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy, \quad t>0$$

<u>Terminology</u>: the function $\dfrac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{\frac{n}{2}}}$ is called the <u>fundamental solution</u> to the heat equation.

Strategy for solving a linear PDE using the Fourier transform

1. Identify the spatial variables of the problem (typically the x-variable for PDEs in x and t. Sometimes a problem is written in $\mathbb{R}^n$ but the x_n-variable should be really thought of as time, e.g. $u_{yy} - u_{xx} = 0$ in $\mathbb{R}^2$ is the wave equation with $y = t$).

2. Apply the Fourier transform with respect to the spatial variable to the equation (and to the initial conditions when they are provided).

3. Typically, the result is either an algebraic equation for $\hat{u}$, or a PDE for $\hat{u}$ with fewer variables (in fact, in many examples it is a ODE for $\hat{u}$).

4. Solve the equation for $\hat{u}$.

5. Find u by the inverse Fourier transform: $u = \mathcal{F}^{-1}(\hat{u})$.