MAT 303 - Summer 10 — Practice Final

REMARK: This practice final covers only part of what you have to know for the final. You should also study carefully the homework problems involving applications from chapter 3 of the textbook and the quizzes that you took in class. Check also the course webpage.

Question 1. Classify the differential equations below as linear or non-linear and state their order.
(a) $y' + \cos(x) = 0$
(b) $\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + x = 10t^2 + t$
(c) $(y'')^3 - y = \frac{1}{x}$
(d) $e^{\sin x^2}\frac{dy}{dx} + xy = e^{-x}$
(e) $7x^2dx + y^3dy = 0$

Question 2. Find the general solution of the following differential equations.
(a) $y'' + 4y = 3\csc t$
(b) $y'' + 4y' + 4y = t^{-2}e^{-2t}$
(c) $y'' - 2y' + y = e^t/(1+t^2)$

Question 3. Verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equation. Then find a particular solution of the non-homogeneous equation.
(a) $t^2y'' - 2y = 3t^2 - 1,\ t > 0, y_1 = t^2,\ y_2 = t^{-1}$.
(b) $(1-x)y'' + xy' - y = \sin x,\ 0 < x < 1, y_1 = e^x,\ y_2 = x$.

Question 4. A mass of $5kg$ stretches a spring $10cm$. The mass is acted on by an external force of $10\sin(t/2)N$ and moves in a medium that imparts a viscous force of $2N$ when the speed of the mass is $4cm/s$. If the mass is set in motion from its equilibrium position with an initial velocity of $3cm/s$:

(a) formulate the initial value problem describing the motion of the mass.
(b) find the solution of the problem.
(c) identify the transient and steady-state parts of the solution.
(d) find the amplitude and phase of the steady-state solution.

Question 5. Consider the two interconnected tanks shown in figure 1. Tank 1 initially contains $30gal$ of water and $25oz$ of salt, while tank 2 initially contains $20gal$ of water and $15oz$ of salt. Water containing $1oz/gal$ of salt flows into tank 1 at a rate of $1.5gal/min$. The mixture flows from tank 1 to tank 2 at a rate of $3gal/min$. Water containing $3oz/gal$ of salt also flows into tank 2 at a rate of $1gal/min$ (from the outside, see picture). The mixture drains from tank 2 at a rate of $4gal/min$, of which some flows back to tank 2 at a rate of $1.5gal/min$, while the remainder leaves the tank.
(a) Let $Q_1(t)$ and $Q_2(t)$, respectively, be the amount of salt in each tank at time t. Write down differential equations and initial conditions that model the flow process. Observe that the system of differential equations is non-homogeneous.
(b) Find the values of $Q_1(t)$ and $Q_2(t)$ for which the system is in equilibrium, i.e., does not change with time. Let Q_1^E and Q_2^E be the equilibrium values. Can you predict which tank will approach its equilibrium state more rapidly?
(c) Let $x_1(t) = Q_1(t) - Q_1^E$ and $x_2(t) = Q_2(t) - Q_2^E$. Determine an initial value problem for x_1 and x_2. Observe that the system of equations for x_1 and x_2 is homogeneous.
(d) Find $Q_1(t)$ and $Q_1(t)$.

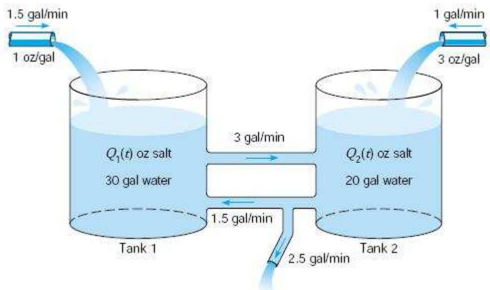

Figure 1: Tanks of problem 5

Question 6. Find the general solution of the given system of differential equations and describe the behavior of the solutions as $t \to \infty$.

(a) $\frac{d\vec{x}}{dt} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \vec{x}$

(b) $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \vec{x}$

(c) $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \vec{x}$

(d) $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -8 & -5 & -5 \end{bmatrix} \vec{x}$

Question 7. In each of the systems below, find the solution by making the transformation $\vec{x} = \vec{u} + \vec{x}_0$, where $\vec{x_0}$ is the inhomogeneous term.

(a) $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \vec{x} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

(b) $\frac{d\vec{x}}{dt} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \vec{x} - \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

(c) $\frac{d\vec{x}}{dt} = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix} \vec{x} - \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

Question 8. Use Laplace transforms to solve the initial value boundary value problems.

(a) $x'' - x' - 2x = 0,\ x(0) = 0,\ x'(0) = 2$

(b) $x'' + x = \cos(3t),\ x(0) = 1,\ x'(0) = 0$

(c) $x'' + 4x = g(t),\ x(0) = 1,\ x'(0) = 0$ and $g(t)$ is defined by $g(t) = 1$ for $0 \le t < \pi$ and $g(t) = 0$ for $\pi \le t < \infty$