VANDERBILT UNIVERSITY
MATH 294 — PARTIAL DIFFERENTIAL EQUATIONS.
HW 5.

Question 1. Let Ω be a bounded domain and consider the problem

$$\begin{cases} \Delta u - u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

where $f \in L^2(\Omega)$ is given. We say that $u \in W_0^{1,2}(\Omega)$ is a *weak solution* of (1) if

$$(u, v)_1 = -(f, v)_0$$

for all $v \in W_0^{1,2}(\Omega)$, where $(\cdot, \cdot)_1$ is the inner product on $W^{1,2}(\Omega)$ and $(\cdot, \cdot)_0$ is the L^2 inner product. Prove that for each f, there exists a unique weak solution u to problem (1). Hint: Riesz representation.

Question 2 (open-ended question). Explain how the above definition of weak solutions is well-motivated by considering the case where f is a sufficiently differentiable function and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ a solution of (1).