VANDERBILT UNIVERSITY
MATH 294 — PARTIAL DIFFERENTIAL EQUATIONS
HW 1

Let Ω be a domain in $\mathbb{R}^n$, i.e., $\Omega \subseteq \mathbb{R}^n$ is an open and connected set contained in $\mathbb{R}^n$. For concreteness you can imagine that Ω is the ball of radius one centered at the origin.

This assignment involves concepts that you learned in previous courses (such as vector spaces, linear maps, etc). If you do not remember them, this is a good chance to refresh your memory since we will be needing some of these concepts later on in the course.

Problem 1. Read chapter 1, and appendix C up to Theorem 3 on page 712 of the textbook. Do problems 1 to 5 of page 12. Some of the concepts introduced in the appendix have not been discussed in class, you may skip them. But if you are curious about it, come to my office hours.

Problem 2. Give an example of a function whose derivative exists but is not continuous.

For problems 3 to 7, recall that we say that a function u is k-times continuously differentiable if all its derivatives up to order k exist and are continuous. Let

$$C^k(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \,\Big|\, u \text{ is } k\text{-times continuously differentiable } \Big\},$$

and

$$C^\infty(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \,\Big|\, u \in C^k(\Omega) \text{ for every } k \Big\}.$$

Problem 3. Show that $C^k(\Omega)$ is a vector space.

Problem 4. Show that the Laplacian Δ is a linear map between $C^k(\Omega)$ and $C^{k-2}(\Omega)$, $k \geq 2$.

Problem 5. Show that $C^\infty(\Omega)$ is a vector space.

Problem 6. Show that the Laplacian Δ is a linear map from $C^\infty(\Omega)$ to itself.

Problem 7. Give a reasonable argument for why $C^k(\Omega)$ is an infinite-dimensional vector space. You are not asked to provide a mathematical and rigorous proof. Instead, you should use your knowledge of calculus and linear algebra to construct a sensible explanation, even if only an intuitive one.

Problem 8. Let $u : \mathbb{R}^4 \to \mathbb{R}$ be the function

$$u(x_1 x_2, x_3, x_4) = e^{x_1^2 + x_2^2}\big(x_3 x_4 - x_1^3 \sin(x_1 x_3)\big).$$

(a) Find $D^2 u(x)$.
(b) Find

$$\sum_{|\alpha| \leq 3} D^\alpha u(x).$$

Extra credit. Prove the integration by parts formula in $\mathbb{R}^2$.