VANDERBILT UNIVERSITY

MATH 2610 – ORDINARY DIFFERENTIAL EQUATIONS

Test 2 – Solutions

NAME: Solutions

Directions. This exam contains six questions and an extra credit question. Make sure you clearly indicate the pages where your solutions are written. Answers without justification will receive little or no credit. Write clearly, legibly, and in a logical fashion. Make precise statements (for instance, write an equal sign if two expressions are equal; say that one expression is a consequence of another when this is the case, etc.).

If you need to use a theorem that was stated in class, you do not need to prove it, unless a question explicitly says so. You do need, however, to state the theorems you invoke.

If you do not understand a question, or think that some problem is ambiguous, missing information, or incorrectly stated, write how you interpret the problem and solve it accordingly.

Question	Points
1 (10 pts)	
2 (15 pts)	
3 (15 pts)	
4 (20 pts)	
5 (20 pts)	
6 (20 pts)	
Extra Credit (10 pts)	
TOTAL (100 pts)	

Question 1. (10 pts) Two tanks, each holding 100 L of liquid, are interconnected by pipes, with the liquid flowing from tank A into tank B at a rate of 3 L/min and from tank B into tank A at a rate of 1 L/min. The liquid inside each tank is kept well-stirred. A brine solution with a concentration of 0.2 kg/L of salt flows from an exterior pipe into tank A at a rate of 6 L/min. The diluted solution flows out of the system from tank A through a second exterior pipe at a rate of 4 L/min, and from tank B at 2 L/min through a third exterior pipe. If, initially, tank A contains pure water and tank B contains 20 kg of salt, write an initial value problem whose solution gives the amount of salt in tanks A and B at time t. You do not have to solve the initial value problem.

Solution 1. Let $x = x(t)$ and $y = y(t)$ be the amount of salt in tanks A and B, respectively. We have $x(0) = 0$ and $y(0) = 20$ kg. Note that the volume on each tank remains constant throughout. The rate of change in x is

$$\begin{aligned}\frac{dx}{dt} &= \text{ in } - \text{ out } = 6\frac{L}{min}0.2\frac{kg}{L} + 1\frac{L}{min}\frac{y}{100}\frac{kg}{L} - 3\frac{L}{min}\frac{x}{100}\frac{kg}{L} - 4\frac{L}{min}\frac{x}{100}\frac{kg}{L}\\ &= 1.2\frac{kg}{min} - \frac{7x}{100}\frac{kg}{min} + \frac{y}{100}\frac{kg}{min}.\end{aligned}$$

The rate of change in y is

$$\begin{aligned}\frac{dy}{dt} &= \text{ in } - \text{ out } = 3\frac{L}{min}\frac{x}{100}\frac{kg}{L} - 2\frac{L}{min}\frac{y}{100}\frac{kg}{L} - 1\frac{L}{min}\frac{y}{100}\frac{kg}{L}\\ &= \frac{3x}{100}\frac{kg}{min} - \frac{3y}{100}\frac{kg}{min}.\end{aligned}$$

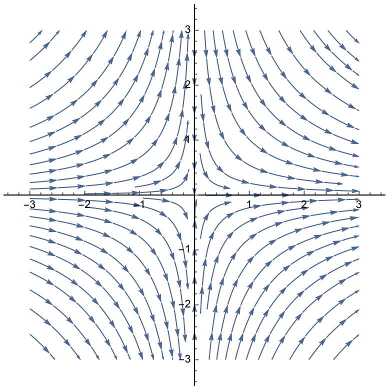
Therefore,

$$\begin{aligned}\frac{dx}{dt} &= 1.2 - \frac{7x}{100} + \frac{y}{100},\\ \frac{dy}{dt} &= \frac{3x}{100} - \frac{3y}{100}\\ x(0) &= 0,\\ y(0) &= 20.\end{aligned}$$

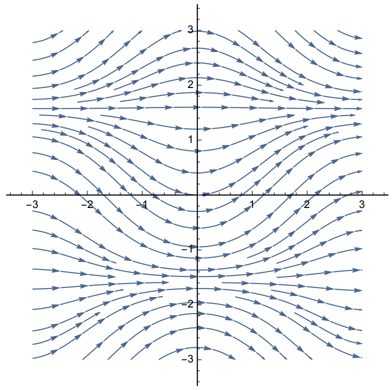
See pages 86-87 of the class notes for interconnected tanks.

Question 2. (15 pts) Match the direction fields with the given differential equations.

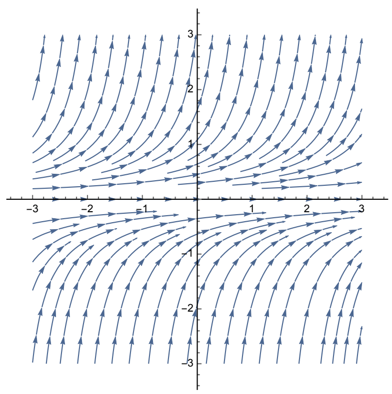
(a) $y' = y^2$ (b) $y' = \cos y \sin x$ (c) $y' = -\dfrac{y}{x}$



(A)



(B)



(C)

Solution 2. In (a), $y' \geq 0$ for any y. The only direction field with non-negative slope everywhere is (C). In (c), $y' \to \pm\infty$ when $x \to 0$. The only direction field with this property is (A). By elimination, (b) corresponds to (B).

Question 3. (15 pts) Give the form of the particular solution for the systems below. You do not have to find the constants involved in the particular solution.

(a) $x' = Ax + f$

where

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 4 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } f(t) = \begin{bmatrix} 1+t \\ t^3 \\ 0 \end{bmatrix}.$$

(b) $x' = Ax + f$

where

$$A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \text{ and } f(t) = \begin{bmatrix} e^t \\ 1 \end{bmatrix}.$$

Hint: $(3,1)$ and $(1,1)$ are eigenvectors of A.

(c) $x' = Ax + f$

where A is a 3×3 matrix with eigenvalues $1 \pm i$ and -1 and

$$f(t) = e^t \begin{bmatrix} t \\ 2 \\ 1 \end{bmatrix}.$$

Solution 3. (a) A is a lower-triangular matrix, thus its eigenvalues are the elements in the diagonal. Since they are distinct, A admits three linearly independent eigenvectors u_1, u_2, and u_3 (page 125 of the class notes). Thus, $x_1(t) = e^{-t}u_1$, $x_2(t) = e^{2t}u_2$, and $x_3(t) = e^t u_3$ are three linearly independent solutions of $x' = Ax$ (page 124 of the class notes). Since f is a polynomial in t and x_1, x_2, and x_3 are not polynomials, the particular solution will have the same form as f (page 135 of the class notes). Therefore

$$x_p(t) = t^3 a + t^2 b + tc + d,$$

where a, b, c, and d are constant vectors. Note that we do not need to find u_1, u_2, or u_3 to answer this question.

(b) To say that $u_1 = (3,1)$ and $u_2 = (1,1)$ are eigenvectors means that $Au_1 = \lambda_1 u_1$ and $Au_2 = \lambda_2 u_2$ (page 123 of the class notes). Thus

$$\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and we conclude that 1 and -1 are eigenvalues of A. The vectors $(3,1)$ and $(1,1)$ are clearly linearly independent (we can also conclude this from the fact that the corresponding eigenvectors are distinct, see page 125 of the class notes). It follows that $x_1(t) = e^t(3,1)$ and $x_2(t) = e^{-t}(1,1)$ are two linearly independent solutions of $x' = Ax$ (page 124 of the class notes). The non-homogeneous term can be written as $e^t(1,0) + (0,1)$, and we see that it repeats the form of x_1. Therefore (page 135 of the class notes and the superposition principle), the form of the particular solution is

$$x_p(t) = te^t a + e^t b + c,$$

where a, b, and c are constant vectors.

(c) There are two linearly independent solutions x_1 and x_2 to $x' = Ax$ which are associated with the complex conjugate eigenvalues; they are combinations of $e^t \cos t$ and $e^t \sin t$ (page 126 of the class notes). The third eigenvalue -1 is real, hence distinct from the previous two, and therefore a third linearly independent solution to the associated homogeneous equation is proportional to e^{-t} (page 124 of the class notes). Because f does not repeat the form of any of such solutions, we have (page 135 of the class notes)

$$x_p(t) = e^t(at + b),$$

where a and b are a constant vectors.

Question 4. (20 pts) Consider the system

$$x' = Ax + f, \tag{1}$$

where A is a $n \times n$ matrix function and f is a vector function.

(a) State the variation of parameters formula for particular solutions of (1).

(b) Give conditions on A and f guaranteeing that the formula you wrote in (a) is well-defined.

(c) Prove the formula you stated in (a).

(d) State conditions on A and f that guarantee that the initial value problem

$$\begin{aligned} x' &= Ax + f, \\ x(t_0) &= x_0, \end{aligned} \tag{2}$$

admits a unique solutions. Then, based on the formula you stated in (a), find a formula for the solution of (2)

Solution 4. (a) The formula is

$$x_p(t) = X(t) \int (X(t))^{-1} f(t)\, dt,$$

where X is a fundamental matrix for the associated homogeneous equation.

(b) The integral has to be well-defined (X^{-1} will always exist because by assumption X is a fundamental matrix, see page 124 of the class notes). A sufficient condition for the integral to make sense is that X^{-1} and f be continuous.

(c) Let X be as above. Set $x_p = Xv$, where v a vector function to be determined. Plugging in:

$$x_p' = (Xv)' = X'v + Xv' = Ax_p + f = AXv + f.$$

Recalling that $X' = AX$, we conclude $Xv' = f$. Solving for v', integrating (without adding a constant of integration) and plugging v back into x_p, yields the result.

(d) A sufficient condition for the existence of a unique solution on an interval containing t_0 is that A and f be continuous (page 116 of the class notes). Such a unique solution can be written as

$$x = Xc + x_p,$$

where c is a constant vector determined by the initial conditions (pages 121 and 122 of the class notes). Using (a), we can write

$$x(t) = X(t)c + X(t) \int_{t_0}^{t} (X(s))^{-1} f(s)\, ds,$$

from which we find $c = X(t_0)^{-1} x_0$, thus

$$x(t) = X(t)X(t_0)^{-1} x_0 + X(t) \int_{t_0}^{t} (X(s))^{-1} f(s)\, ds.$$

All items in these question can be found on pages 136-137 of the class notes.

Question 5. (20 pts) Consider the system $x' = Ax$, where

$$A = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}.$$

To answer the questions that follow, you are allowed to use the following facts:

(i) The matrix A has eigenvalue -1 with multiplicity two.

(ii) A possesses only one linearly independent eigenvector, which can be taken to be $(1, 2)$.

(iii) We have

$$A = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(iv) We have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

(a) Find a fundamental matrix for the given system.

(b) Using the fundamental matrix you found in (a), compute e^{At}.

Solution 5. Because A has only one eigenvalue, -1, and only one linearly independent corresponding eigenvector, the eigenvectors of A produce only one solution given by eigenvectors, which we can take to be $x_1(t) = e^{-t}(1, 2)$. To find a second linearly independent solution, we need to calculate the generalized eigenvectors of A. Compute

$$A - \lambda I = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$$

and we need to solve $(A - \lambda I)^2 u = 0$, i.e.,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Any u will be a solution, but since we want a generalized eigenvector, we need to choose u non-zero, and because we want a second linearly independent solution, u needs to be linearly independent from $(1, 2)$. We can take, e.g., $(1, 0)$. Then

$$\begin{aligned} x_2(t) &= e^{-t}(u + t(A - (-1)I)u) \\ &= e^{-t} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \\ &= e^{-t} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right) \\ &= e^{-t} \begin{bmatrix} 1 + 2t \\ 4t \end{bmatrix}. \end{aligned}$$

Therefore, a fundamental matrix is

$$X(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix} = e^{-t} \begin{bmatrix} 1 & 1 + 2t \\ 2 & 4t \end{bmatrix}$$

See pages 142-144 of the class notes for the above procedure.

(b) We have $e^{At} = X(t)(X(0))^{-1}$ (page 141 of the class notes), thus

$$e^{At} = \begin{bmatrix} 1 & 1+2t \\ 2 & 4t \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^{-1} = -\frac{1}{2}e^{-t} \begin{bmatrix} 1 & 1+2t \\ 2 & 4t \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2t & -t \\ 4t & 1-2t \end{bmatrix}.$$

Question 6. (20 pts) True or false?
(a) Let $x_1, \cdots, x_n$ be n-component vector functions defined on an interval I such that $W[x_1, \ldots, x_n](t_0) \neq 0$, where $t_0 \in I$ and $W[x_1, \ldots, x_n](t_0)$ is the Wronskian of $x_1, \ldots, x_n$ at t_0. Then $x_1, \ldots, x_n$ are linearly independent.
(b) Consider the differential equation $y'' + py' + qy = f$, where p, q, and f are continuous functions defined on an interval I. Let y_p be a particular solution and let y_1 and y_2 be two linearly independent solutions of the associated homogeneous equation, where y_p, y_1, and y_2 are defined on I. Let Y be defined on I and satisfy $Y'' + pY' + qY = f$. Then the functions Y, y_1, y_2, and y_p are linearly dependent.
(c) Let A be a $n \times n$ matrix and u a generalized eigenvector of A associated to an eigenvalue λ. Then $e^{\lambda t} u$ is a solution to $x' = Ax$.
(d) Consider the differential equation $y'' + py' + qy = 0$, where p and q are continuous functions defined on an interval I, and let y_1 be a solution. Then the function y_2 given by

$$y_2(t) = y_1(t) \int \frac{e^{-\int p(t)\,dt}}{(y_1(t))^2}\,dt$$

is a second, linearly independent solution to the differential equation.

Solution 6. (a) True, this is precisely the statement of one of the theorems proven in class (page 119 of the class notes).

(b) True. The function Y can be written as (page 77 of the class notes) $Y = c_1 y_1 + c_2 y_2 + y_p$ for some constants c_1 and c_2, thus the functions are linearly dependent.

(c) False. For example, taking the generalized vector $(1, 0)$ from question 5, we see that

$$(e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix})' = -e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix} e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(See the remark on page 146 of the class notes.)

(d) False. The statement would be true if one assumes that y_1 is not identically zero, otherwise the given expression might not even be well-defined (see page 79 of the class notes).

Extra credit. (10 pts) Let A be a $n \times n$ matrix.

(a) State the definition of e^A.

(b) Prove that e^{At} is a fundamental matrix for the system $x' = Ax$.

Solution 7. (a) The definition is

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!},$$

where $A^0 = I$.

(b) (This was done on page 141 of the class notes.) We have

$$\frac{d}{dt}e^{At} = \frac{d}{dt}\sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = \sum_{k=0}^{\infty} k\frac{A^k t^{k-1}}{k!} = A\sum_{k=1}^{\infty} \frac{A^{k-1}t^{k-1}}{(k-1)!} = A\sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = Ae^{At},$$

thus e^{At} satisfies the matrix differential equation $X' = AX$. Because e^{At} is invertible (page 140 of the class notes), e^{At} is a fundamental matrix.