VANDERBILT UNIVERSITY

MATH 2610 – ORDINARY DIFFERENTIAL EQUATIONS

Test 1 – Solutions

NAME: Solutions.

Directions. This exam contains seven questions and an extra credit question. Make sure you clearly indicate the pages where your solutions are written. Answers without justification will receive little or no credit. Write clearly, legibly, and in a logical fashion. Make precise statements (for instance, write an equal sign if two expressions are equal; say that one expression is a consequence of another when this is the case, etc.).

If you need to use a theorem that was stated in class, you do not need to prove it, unless a question explicitly says so. You do need, however, to state the theorems you invoke.

If you do not understand a question, or think that some problem is ambiguous, missing information, or incorrectly stated, write how you interpret the problem and solve it accordingly.

Question	Points
1 (10 pts)	
2 (10 pts)	
3 (15 pts)	
4 (20 pts)	
5 (20 pts)	
6 (20 pts)	
7 (05 pts)	
Extra Credit (10 pts)	
TOTAL (100 pts)	

Question 1. (10 pts) For each equation below, identify the unknown function, classify the equation as linear or non-linear, and state its order.

(a) $\sqrt{y}\dfrac{dy}{dx} + x^2 y = 0.$

(b) $u'' + u = \cos x$

(c) $x''' = -\sin x\, x'.$

Solution 1. (a) Unknown: y. Non-linear. First order.
(b) Unknown: u. Linear. Second order.
(c) Unknown: x. Non-linear. Third order.

Question 2. (10 pts) Consider a tank holding 100 gal of brine with a concentration of 2 lb of salt per gallon. A solution of salt begins to flow at a constant rate of 5 gal/min. The solution inside the tank is kept well stirred and is flowing out of the tank at a rate of 5 gal/min. The concentration of salt in the brine entering the tank is 3 lb per gal.

(a) Find an initial value problem whose solution gives the amount of salt in the tank at time t.

(b) What is the maximum concentration that can be reached inside the tank?

Solution 2. (a) Denote by $x(t)$ the amount of salt in the tank at time t. We have

$$\begin{aligned}\frac{dx}{dt} &= \text{in} \ - \ \text{out} \\ &= 5\frac{gal}{min}\,3\frac{lb}{gal} - 5\frac{gal}{min}\,\frac{x}{100}\frac{lb}{gal} \\ &= 15\frac{lb}{min} - \frac{5}{100}x\frac{lb}{min.}\end{aligned}$$

Since $x(0) = 2(lb/gal)\,100gal$, we have

$$\begin{aligned}\frac{dx}{dt} + \frac{5}{100}x &= 15, \\ x(0) &= 200.\end{aligned}$$

(b) The equation is a first order linear differential equation with $P(t) = 5/100$ and $Q(t) = 15$. We will use the formula

$$x(t) = e^{-\int P(t)\,dt}(\int Q(t)e^{\int P(t)\,dt}\,dt + C),$$

where C is an arbitrary constant.

We find

$$\begin{aligned}x(t) &= e^{-\int \frac{5}{100}\,dt}(\int 15e^{\int \frac{5}{100}\,dt}\,dt + C) \\ &= e^{-\frac{5}{100}t}(15\int e^{\frac{5}{100}t}\,dt + C) \\ &= e^{-\frac{5}{100}t}(\frac{15 \times 100}{5}e^{\frac{5}{100}t} + C) \\ &= 300 + Ce^{-\frac{5}{100}t}.\end{aligned}$$

To determine C, we use the initial condition

$$x(0) = 200 = 300 + Ce^0 \Rightarrow C = -100.$$

Thus

$$x(t) = 300 - 100e^{-\frac{5}{100}t}.$$

Because $e^{-\frac{5}{100}t}$ is a decreasing function, $x(t)$ increases with t and is asymptotic to

$$\lim_{t\to\infty} x(t) = 300.$$

Thus the maximum concentration is $300lb/(100gal) = 3lb/gal$.

Question 3. (15 pts) Find the general solution of the given differential equation.
(a) $x' - x^2 = 0$.

(b) $x'' + 5x' + 6x = e^{2t}$.

(c) $y' = -\dfrac{4x^3 + y}{4y^3 + x}$.

Solution 3. (a) This is a separable equation. For $x \neq 0$

$$\frac{dx}{x^2} = dt \Rightarrow \frac{1}{x} = -t + C \Rightarrow x = \frac{1}{C - t},$$

where C is an arbitrary constant. We immediately verify that $x = 0$ is also a solution, hence the general solution is $x = 1/(C - t)$ or $x = 0$.

(b) This is a linear second order inhomogeneous equation. The characteristic equation is $\lambda^2 + 5\lambda + 6 = (\lambda + 2)(\lambda + 3) = 0$. Hence $x_h = c_1 e^{-2t} + c_2 e^{-3t}$, where c_1 and c_2 are arbitrary constants, is the general solution of the associated homogeneous equation. Since this does not contain e^{2t}, we seek a particular solution in the form $x_p = Ae^{2t}$, A constant. Plugging in produces

$$4A + 10A + 6A = 1 \Rightarrow A = \frac{1}{20}.$$

The general solution is $x = c_1 e^{-2t} + c_2 e^{-3t} + \frac{1}{20} e^{2t}$.

(c) Write the equation as

$$(4x^3 + y)\, dx + (4y^3 + x)\, dy = M\, dx + N\, dy = 0.$$

Then, $\partial_y M = 1 = \partial_x N$ and this is an exact equation. Set

$$F(x, y) = \int (4x^3 + y)\, dx = x^4 + xy + g(y).$$

Next,

$$\partial_y F = x + g'(y) = N = x + 4y^3 \Rightarrow g'(t) = 4y^3 \Rightarrow g(y) = y^4.$$

The general solution is thus

$$x^4 + xy + y^4 = C,$$

where C is an arbitrary constant.

Question 4. (20 pts) Give the form of the particular solution for the given differential equations. You do not have to find the values of the constants of the particular solution.

(a) $x'' - 3x' + 2x = e^{2t}$.

(b) $x'' + 9x = \sin t$.

(c) $x'' - x = 3t^2 + 1$.

(d) $x'' + x' - 2x = e^{-2t} + e^t$.

Solution 4. (a) The characteristic equation is $(\lambda - 1)(\lambda - 2) = 0$, so e^t and e^{2t} are two linearly independent solutions of the associated homogeneous equation. Since the inhomogeneous term repeats one of these solutions, we have

$$x_p = Ate^{2t},$$

where A is a constant.

(b) The characteristic equation is $\lambda^2 + 9 = 0$, so $\cos(3t)$ and $\sin(3t)$ are two linearly independent solutions of the associated homogeneous equation. Since the inhomogeneous term does not repeat either of these solutions, we have

$$x_p = A\cos t + B\sin t,$$

where A and B are constants.

(c) The characteristic equation is $\lambda^2 - 1 = 0$, so e^t and e^{-t} are two linearly independent solutions of the associated homogeneous equation. Since the inhomogeneous term does not repeat either of these solutions, we have

$$x_p = At^2 + Bt + C,$$

where A, B, and C are constants.

(d) The characteristic equation is $(\lambda + 2)(\lambda - 1) = 0$, so e^{-2t} and e^t are two linearly independent solutions of the associated homogeneous equation. Each inhomogeneous term repeats one of these solutions. Thus, we have

$$x_p = Ate^{-2t} + Bte^t,$$

where A and B are constants.

Question 5. (20 pts) Consider the following initial value problem:

$$y' - \sqrt{y} - xy^2 = 0,$$
$$y(1) = a.$$

Determine for which values of a this problem admits a unique solution.

Solution 5. Write the equations as $y' = f(x, y)$, with $f(x, y) = \sqrt{y} + xy^2$. Note that this problem is not defined for $y < 0$.

By the the existence and uniqueness theorem for first order equations seen in class, a solution satisfying $y(x_0) = y_0$ will exist and be unique in a neighborhood of $x = x_0$ if $\partial_y f$ exists and is continuous in a neighborhood of (x_0, y_0).

In our case, $\partial_y f(x, y) = \frac{1}{2}\frac{1}{\sqrt{y}} + 2xy$, which is continuous and well-defined for all x and all $y > 0$. Hence, the given initial value problem admits a unique solution if $a > 0$.

Question 6. (20 pts) True or false? Justify your answers.

(a) If $p(x)$ and $q(x)$ are continuous functions on the interval (a, b), then the initial value problem

$$y'(x) + p(x)y(x) = q(x),$$
$$y(x_0) = y_0,$$

always admits a unique solution for any given $x_0 \in (a, b)$ and $y_0 \in \mathbb{R}$.

(b) Given the equation

$$M(x, y)\, dx + N(x, y)\, dy = 0,$$

it is always possible to find a function $F = F(x, y)$ such that $\frac{\partial F}{\partial x} = M$, $\frac{\partial F}{\partial y} = N$, and the general solution of the differential equation is given by $F(x, y) = C$, where C is an arbitrary constant.

(c) If a, b, and c are constants and $a \neq 0$, the equation

$$ax'' + bx' + cx = 0,$$

always admits two linearly independent solutions $x_1(t)$ and $x_2(t)$ that are defined for all $t \in \mathbb{R}$.

(d) If x_1 and x_2 are two functions such that their Wronskian vanishes, then they are linearly dependent.

Solution 6. (a) True. This follows from the existence and uniqueness theorem for linear first order equations seen in class.

(b) False. According to the study of such equations developed in class, the statement is true if $\partial_y M = \partial_x N$.

(c) True. In class, we established that if λ_1 and λ_2 are the two roots of the characteristic equation, then the two linearly independent solutions are given by $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ if $\lambda_1 \neq \lambda_2$ are real, $e^{\lambda t}$ and $te^{\lambda t}$ if $\lambda_1 = \lambda_2 = \lambda$, and $e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$ if $\lambda_1 = \alpha + i\beta$, $\beta \neq 0$.

(d) False. This is true if x_1 and x_2 are also solutions to a second order linear differential equation.

Question 7. (05 pts) State and prove the superposition principle for second order linear differential equations with constant coefficients.

Solution 7. Statement: if x_1 and x_2 are solutions to $ax'' + bx' + cx = f_1$ and $ax'' + bx' + cx = f_2$, respectively, where a, b, c are constants and $a \neq 0$, then x_1+x_2 is a solution to $ax''+bx'+cx = f_1+f_2$.

Proof: Plug $x_1 + x_2$ into $ax'' + bx' + cx$ to find

$$\begin{aligned} a(x_1 + x_2)'' + b(x_1 + x_2)' + c(x_1 + x_2) &= ax_1'' + bx_1' + cx_1 + ax_2'' + bx_2' + cx_2 \\ &= f_1 + f_2. \end{aligned}$$

Extra credit question. (10 pts) Let $M = M(x, y)$ and $N = N(x, y)$ be two functions such that their partial derivatives exist and are continuous in a rectangle $R \subseteq \mathbb{R}^2$. Prove that the differential equation

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

is exact if and only if the compatibility condition

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$$

holds for every $(x, y) \in R$.

Solution to the extra credit. See the class notes.