VANDERBILT UNIVERSITY

MATH 2610 – ORDINARY DIFFERENTIAL EQUATIONS

Practice for test 2

The second test will cover all material discussed from (including) section 4.6 to (including) section 9.8, with the exception of the Cauchy-Euler equation (i.e., Cauchy-Euler will not be on the test), plus sections 1.3 and 1.4.

Question 1. Consider the equation

$$x^2y'' - 2y = 0,\ x > 0.$$

The functions $y_1 = x^2$ and $y_2 = x^{-1}$ are solutions of the differential equation (you do not have to show this). Are y_1 and y_2 linearly independent?

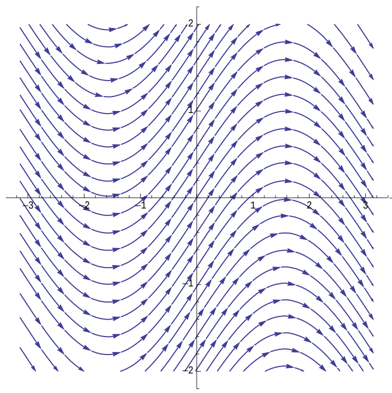
Solution 1. Because y_1 and y_2 are solutions to the equations, linear dependence/independence can be decided by the Wronskian:

$$\begin{aligned} W(y_1, y_2)(x) &= y_1y_2' - y_1'y_2 \\ &= x^2(-x^{-2}) - 2xx^{-1} \\ &= -3 \neq 0. \end{aligned}$$

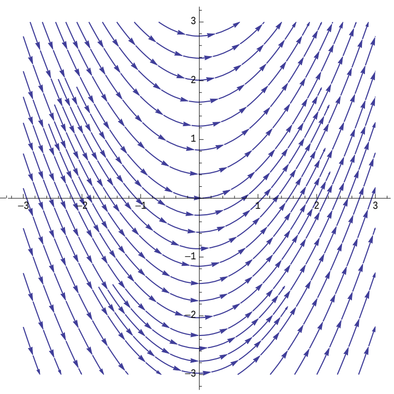
Therefore, y_1 and y_2 are linearly independent on $(0, \infty)$.

Question 2. Match the direction fields with the given differential equations.

(a) $y' = -\dfrac{y}{x}$ (b) $y' = \cos x$ (c) $y' = y(1 - 0.5y)$ (d) $y' = x$



(A)



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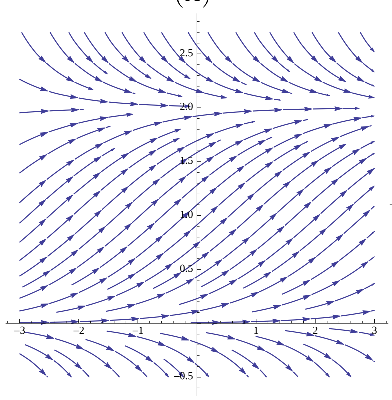
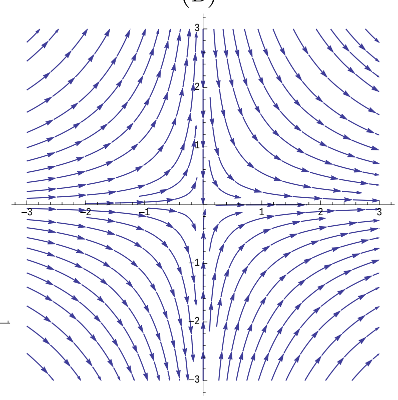


(C)



(D)

Solution 2. (a) = D, (b) = A, (c) = C, (d) = B.

Question 3. For the systems $x' = Ax + f$ given below:

(a) Find the general solution if

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } f(t) = e^{-2t} \begin{bmatrix} t \\ 3 \end{bmatrix}.$$

(b) State the form of the particular solution if

$$A = \begin{bmatrix} -1 & 0 \\ 2 & 2 \end{bmatrix} \text{ and } f(t) = \begin{bmatrix} t^2 \\ t+1 \end{bmatrix}.$$

Solution 3. (a) First we solve $x' = Ax$. The eigenvalues of A are 1 and 2, and corresponding eigenvectors are $(1, 0)$ and $(1, 1)$. Thus,

$$x_h = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The non-homogeneous term does not repeat any term of x_h, thus we seek a particular solution of the same form as f, i.e., $x_p = e^{-2t}(ta + b)$, where a and b are two-component vectors to be determined. Plugging in and carrying out some calculations, we find

$$x_p = e^{-2t} \begin{bmatrix} -(1/3)t + 5/36 \\ -3/4 \end{bmatrix}.$$

and the general solution is $x = x_h + x_p$.

(b) The eigenvalues of A are -1 and 2. The non-homogeneous term will not repeat any term of the associated homogeneous equation and therefore

$$x_p = t^2 a + tb + c,$$

where a, b, and c are two-component vectors.

Question 4. Determine e^A if

$$A = \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix}.$$

Solution 4. To find e^A, we first compute e^{At} and then plug $t = 1$. e^{At}, in turn, can be computed by finding a fundamental matrix to the system $x' = Ax$.

Compute

$$\det \begin{bmatrix} 5-\lambda & -4 & 0 \\ 1 & -\lambda & 2 \\ 0 & 2 & 5-\lambda \end{bmatrix} = -\lambda(\lambda-5)^2,$$

so $\lambda_1 = 0$ and $\lambda_2 = 5$ are the eigenvalues, with λ_2 of multiplicity two.

To find an eigenvector associated with λ_1, we solve

$$\begin{bmatrix} 5 & -4 & 0 & \vdots & 0 \\ 1 & 0 & 2 & \vdots & 0 \\ 0 & 2 & 5 & \vdots & 0 \end{bmatrix}.$$

Applying Gauss-Jordan elimination we find $u_1 = (-4, -5, 2)$, and $x_1 = e^{0t}u_1 = (-4, -5, 2)$ is a solution to $x' = Ax$.

Next, we move to λ_2, and consider:

$$\begin{bmatrix} 0 & -4 & 0 & \vdots & 0 \\ 1 & -5 & 2 & \vdots & 0 \\ 0 & 2 & 0 & \vdots & 0 \end{bmatrix}.$$

Applying Gauss-Jordan elimination, we find

$$\begin{bmatrix} 1 & 0 & 2 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Thus, this system has only one free variable, yielding only one linearly independent eigenvector which we can take to be $u_2 = (-2, 0, 1)$. Hence $x_2 = e^{5t}(-2, 0, 1)$ is a second linearly independent solution to $x' = Ax$. To find a third linearly independent solution, we need to find a generalized eigenvector associated with $\lambda_2 = 5$. Compute

$$(A - 5I)^2 = \begin{bmatrix} 0 & -4 & 0 \\ 1 & -5 & 2 \\ 0 & 2 & 0 \end{bmatrix}^2 = \begin{bmatrix} -4 & 20 & -8 \\ -5 & 25 & -10 \\ 2 & -10 & 4 \end{bmatrix}.$$

Now we solve

$$\begin{bmatrix} -4 & 20 & -8 & \vdots & 0 \\ -5 & 25 & -10 & \vdots & 0 \\ 2 & -10 & 4 & \vdots & 0 \end{bmatrix}.$$

Applying Gauss-Jordan elimination gives

$$\left[\begin{array}{ccc:c} -1 & 5 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which has two free variables that yield two linearly independent generalized eigenvectors $u_2 = (-2, 0, 1)$ and $u_3 = (5, 1, 0)$ (notice that we already knew from above that u_2 is a solution since it is an eigenvector). To find a third (linearly independent) solution to $x' = Ax$, compute

$$x_3 = e^{At}u_3 = e^{5t}(u_3 + t(A - 5I)u_3) = e^{5t}\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + te^{5t}\begin{bmatrix} 0 & -4 & 0 \\ 1 & -5 & 2 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = e^{5t}\begin{bmatrix} 5 - 4t \\ 1 \\ 2t \end{bmatrix}.$$

A fundamental matrix is now given by $X = [x_1\, x_2\, x_3]$, i.e.,

$$X(t) = \begin{bmatrix} -4 & -2e^{5t} & e^{5t}(5 - 4t) \\ -5 & 0 & e^{5t} \\ 2 & e^{5t} & 2e^{5t}t \end{bmatrix}.$$

Recall that $e^{At} = X(t)(X(0))^{-1}$. Plugging $t = 0$ into $X(t)$ and calculating the inverse, we find

$$(X(0))^{-1} = \frac{1}{25}\begin{bmatrix} 1 & -5 & 2 \\ -2 & 10 & 21 \\ 5 & 0 & 10 \end{bmatrix}.$$

Thus,

$$\begin{aligned} e^{At} = X(t)(X(0))^{-1} &= \frac{1}{25}\begin{bmatrix} -4 & -2e^{5t} & e^{5t}(5 - 4t) \\ -5 & 0 & e^{5t} \\ 2 & e^{5t} & 2e^{5t}t \end{bmatrix}\begin{bmatrix} 1 & -5 & 2 \\ -2 & 10 & 21 \\ 5 & 0 & 10 \end{bmatrix} \\ &= \frac{1}{25}\begin{bmatrix} -4 + 29e^{5t} - 20te^{5t} & 20 - 20e^{5t} & -8 + 8e^{5t} - 40te^{5t} \\ -5 + 5e^{5t} & 25 & -10 + 10e^{5t} \\ 2 - 2e^{5t} + 10te^{5t} & -10 + 10e^{5t} & 4 + 21e^{5t} + 20te^{5t} \end{bmatrix}. \end{aligned}$$

Plugging $t = 1$ yields the answer.

Question 5. (a) Let $x_1, \cdots, x_k$ be vector functions defined on an interval I. State what it means for $x_1, \cdots, x_k$ to be linearly independent on I.

(b) Are $(e^t, -e^t)$ and $(5e^t, e^t)$ linearly independent on $(-\infty, \infty)$?

(c) Give an example of vector functions that are linearly independent on $(-\infty, \infty)$ but are linearly dependent on $(0, \infty)$.

Solution 5. (a) $x_1, \cdots, x_k$ are linearly independent on I if the following condition holds. Let $c_1, \ldots, c_k$ be constants such that $c_1 x_1(t) + \cdots + c_k x_k(t) = 0$ for all $t \in I$. Then $c_1 = \cdots = c_k = 0$.

(b) If $c_1(e^t, e^{-t}) + c_2(5e^t, e^t) = 0$ on $(-\infty, \infty)$ then this holds in particular at $t = 0$, thus $c_1(1,1) + c_2(5,1) = 0$, which implies $c_1 = c_2 = 0$.

(c) $(t, |t|)$ and $(|t|, t)$ (see the class notes).

Question 6. (a) Let $x_1, \cdots, x_k$ be vector functions defined on an interval I. State the definition of the Wronskian of $x_1, \cdots, x_k$.

(b) Prove that if the Wronskian of $x_1, \cdots, x_k$ does not vanish at a point $t_0 \in I$, then $x_1, \cdots, x_k$ are linearly independent on I.

Solution 6. This was done in class, see the class notes.

Question 7. Let A be a $n \times n$ continuous matrix function.

(a) What is a fundamental matrix for the system $x' = Ax$?

(b) If X is a fundamental matrix for $x' = Ax$, show that $X' = AX$.

(c) Let X and Y be two fundamental matrices for $x' = Ax$. Show that there exists a constant matrix M such that $X = YM$.

Solution 7. This was done in class, see the class notes.

Question 8. Let A be a $n \times n$ continuous matrix function and f be a continuous vector function, both defined on an interval I.

(a) State the variation of parameters formula for a particular solution to the system $x' = Ax + f$.

(b) Prove the formula you stated in (a).

Solution 8. This was done in class, see the class notes.

Question 9. True or false?

(a) Every $n \times n$ matrix of real numbers has n linearly independent eigenvectors.

(b) If A is a $n \times n$ matrix of real numbers and x is a vector function that satisfies the initial value problem $x' = Ax$, $x(0) = 0$, then $x(t) = 0$ for all t.

(c) If A is a 3×3 matrix of real numbers whose eigenvalues are 2 with multiplicity two and 1, then A has two linearly independent generalized eigenvectors that correspond to the eigenvalue 2.

(d) The Wronskian of n linearly independent vector functions on an open interval I is never zero on I.

(e) If a 3×3 matrix of real numbers has eigenvalues $2+3i$, $2-3i$, and 5, then the matrix has three linearly independent eigenvectors.

Solution 9. (a) False. The matrix

$$A = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}$$

possess only $(1, 2)$ as linearly independent eigenvector.
(b) True, by one of the existence and uniqueness theorems.
(c) True, a matrix always admits a complete set of generalized eigenvectors (see the class notes).
(d) False. The Wronskian of $(1, t)$ and $(1, 2t)$ vanishes at $t = 0$ but the two functions are linearly independent on $(-1, 1)$.
(e) True. Two linearly independent eigenvectors come from the complex conjugate roots $3 \pm 3i$. An eigenvector associated with 5 will be linearly independent from the previous two because eigenvectors associated with distinct eigenvalues are linearly independent.

Question 10. Know the theorems and definitions stated in class. Be prepared to state and use any of the theorems discussed in class, and to prove any theorem that has been proved in class or in an exercise. Also review the homework problems and posted examples.

Solution 10. N/A.