VANDERBILT UNIVERSITY

MATH 2610 – ORDINARY DIFFERENTIAL EQUATIONS

Practice for test 1 — solutions

The first test will cover all material discussed up to (including) section 4.5.

Important: The solutions below are intended to help students with the practice test. They do not provide a model of how answers should be given in the test. In the test you should fully justify your answers and present details of your calculations, whereas here concise answers, often omitting calculations, are given.

Question 1. For each equation below, identify the unknown function, classify the equation as linear or non-linear, and state its order.

(a) $y\dfrac{dy}{dx} + \dfrac{y}{x} = 0$.

(b) $x'''' + \cos t\, x' = \sin t$.

(c) $y''' = -\cos y\, y'$.

Solution 1. (a) Unknown: y. Non-linear, first order. (b) Unknown: x. Linear, fourth order. (c) Unknown: y. Non-linear, third order.

Question 2. Solve the following initial value problems.

(a) $y' = \dfrac{y-1}{x+3}$, $y(-1) = 0$.

(b) $x' = e^{-t} - 4x$, $x(0) = \dfrac{4}{3}$.

Solution 2. (a) This equation is separable, so

$$\frac{dy}{y-1} = \frac{dx}{x+3} \Rightarrow \int \frac{dy}{y-1} = \int \frac{dx}{x+3},$$

from what we obtain

$$|y-1| = C|x+3|,$$

or yet

$$y = 1 + C(x+3).$$

Using the initial condition we find $C = -\frac{1}{2}$, thus

$$y = 1 - \frac{1}{2}(x+3).$$

(b) This is a linear equation with $p(t) = 4$ and $q(t) = e^{-t}$. Using the formula for first order linear equations, we find

$$e^{\int p(t)\, dt} = e^{4t},$$

so that

$$x = \frac{1}{3}e^{-t} + Ce^{-4t}.$$

The initial condition gives $C = 1$, so

$$x = \frac{1}{3}e^{-t} + e^{-4t}.$$

Question 3. Solve the following differential equations.

(a) $y' = \dfrac{\cos y \cos x + 2x}{\sin y \sin x + 2y}$.

(b) $x' = 2t^{-1}x + t^2 \cos t,\ t > 0$.

(c) $x^2 y' = y - 1$.

Solution 3. (a) Write

$$(\cos y \cos x + 2x)dx - (\sin y \sin x + 2y)dy = 0.$$

We readily verify that this equation is exact, with $M(x, y) = \cos y \cos x + 2x$ and $N(x, y) = -(\sin y \sin x + 2y)$. Then

$$F(x, y) = \int M(x, y)\, dx = \sin x \cos y + x^2 + g(y).$$

From

$$\frac{\partial F}{\partial y} = N,$$

we find

$$g'(y) = -2y,$$

hence $g(y) = -y^2$. The general solution is

$$F(x, y) = \sin x \cos y + x^2 - y^2 = C.$$

(b) This is a linear equation with $p(t) = -\frac{2}{t}$, and $q(t) = t^2 \cos t$. Using the formula for first order equations, we find

$$x = t^2 \sin t + Ct^2.$$

(c) This is a separable equation:

$$\frac{dy}{y - 1} = \frac{dx}{x^2}.$$

Integrating, we get

$$\ln|y - 1| = -\frac{1}{x} + C,$$

which leads to

$$y = Ce^{-x^{-1}} + 1.$$

The solution $y = 1$ is included in the above family upon taking $C = 0$.

Question 4. Consider a large tank holding 1000 L of pure water into which a brine solution of salt begins to flow at a constant rate of 6 L/min. The solution inside the tank is kept well stirred and is flowing out of the tank at a rate of 6 L/min. The concentration of salt in the brine entering the tank is 0.1 kg/L.

(a) Find an initial value problem whose solution gives the amount of salt in the tank at time t.

(b) Solve the initial value problem in (a).

(c) When will the concentration in the tank reach 0.05 kg/L?

Solution 4. (a) Let $x(t)$ be the amount of salt in the tank at time t. Then

$$\begin{aligned}\frac{dx}{dt} &= \text{in} - \text{out} \\ &= 6\frac{L}{min}0.1\frac{kg}{L} - 6\frac{L}{min}\frac{x}{1000L} \\ &= (0.6 - \frac{3x}{500})\frac{kg}{min}.\end{aligned}$$

Because $x(0) = 0$, the IVP is

$$\begin{aligned}\frac{dx}{dt} + \frac{3x}{500} &= 0.6, \\ x(0) &= 0.\end{aligned}$$

(b) Using the formula for first order linear equations and the initial condition we find

$$x(t) = 100(1 - e^{-\frac{3t}{500}}),$$

(c) The concentration at time t is given by

$$\frac{x(t)}{1000} = 0.1(1 - e^{-\frac{3t}{500}}),$$

so that

$$0.05 = 0.1(1 - e^{-\frac{3t}{500}}) \Rightarrow t = \frac{500 \ln 2}{3}.$$

Question 5. Find the general solution of the given differential equation.

(a) $x'' + 8x' - 14x = 0$.

(b) $x'' + 8x' - 9x = 0$.

Solution 5. (a) $\lambda^2 + 8\lambda - 14 = 0 \Rightarrow \lambda = -4 \pm \sqrt{30}$. $x = c_1 e^{(-4+\sqrt{30})t} + c_2 e^{(-4-\sqrt{30})t}$.

(b) $\lambda^2 + 8\lambda - 9 = 0 \Rightarrow \lambda = -9, \lambda = 1$. $x = c_1 e^{-9t} + c_2 e^{t}$.

Question 6. Give the form of the particular solution for the given differential equations. You do not have to find the values of the constants of the particular solution.

(a) $x'' + 2x' - 3x = \cos t$.

(b) $x'' + 4x = 8\sin 2t$.

(c) $x'' - 2x' + x = e^t \cos t$.

(d) $x'' - x' - 12x = 2t^6 e^{-3t}$.

Solution 6. (a) The characteristic equation is

$$\lambda^2 + 2\lambda - 3 = (\lambda - 1)(\lambda + 3) = 0.$$

Hence $x_1 = e^t$ and $x_2 = e^{-3t}$ are linearly independent solutions of the associated homogeneous equation. Since these do not involve $\cos t$, we have

$$x_p = A\cos t + B\sin t.$$

(b) The characteristic equation is

$$\lambda^2 + 4 = 0.$$

Hence $x_1 = \cos 2t$ and $x_2 = \sin 2t$ are linearly independent solutions of the associated homogeneous equation. Therefore we need to take $s = 1$, so

$$x_p = At\cos t + Bt\sin t.$$

(c) The characteristic equation is

$$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0.$$

Hence $x_1 = e^t$ and $x_2 = te^t$ are linearly independent solutions of the associated homogeneous equation. Thus,

$$x_p = (A\cos t + B\sin t)e^t.$$

(d) The characteristic equation is

$$\lambda^2 - \lambda - 12 = (\lambda + 3)(\lambda - 4) = 0.$$

Hence $x_1 = e^{-3t}$ and $x_2 = e^{4t}$ are linearly independent solutions of the associated homogeneous equation. We need to take $s = 1$, thus

$$x_p = t(a_6t^6 + a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_2t^1 + a_0)e^{-3t}.$$

Question 7. Verify that the given functions are two linearly independent solutions of the differential equation.

(a) $x^2y'' - 2y = 0$, $x > 0$, $y_1 = x^2$, $y_2 = x^{-1}$.

(b) $(1 - x)y'' + xy' - y = 0$, $0 < x < 1$, $y_1 = e^x$, $y_2 = x$.

Solution 7. The verification is done by plugging in the given functions into the equation, while linear independence is verified using the Wronskian.

Question 8. Show that the problem

$$3x' - t^2 + tx^3 = 0,\ x(1) = 6,$$

has a unique solution defined in some neighborhood of $t = 1$.

Solution 8. Write

$$x' = f(t, x),\ x(1) = 6,$$

where $f(t, x) = \frac{t^2 - tx^3}{3}$. Since $f(t, x)$ and $\partial_x f(t, x) = -tx^2$ are continuous in the neighborhood of $(1, 6)$, the result follows from the existence and uniqueness theorem for first order equations.

Question 9. Review the homework problems and examples posted in the course webpage.

Question 10. Know the statement, proof, and how to use the theorems established in class.