VANDERBILT UNIVERSITY

MATH 2610 – ORDINARY DIFFERENTIAL EQUATIONS

Practice for the final test – solutions

The final exam will be cumulative with an emphasis on chapter 12. The test will cover all material, except for Cauchy-Euler equations in section 4.7, sections 1.3, 1.4, 5.2, 5.3, non-conservative systems in section 12.4, and section 12.7.

Question 1. Consider the IVP $y' = f(x, y)$, $y'(x_0) = y_0$. State a theorem that guarantees the existence of a unique solution defined in a neighborhood of x_0. Using the theorem that you stated, show that $y' = (\sin y)^{-1}$, $y(0) = \pi/4$ admits a unique solution in some neighborhood of 0.

Solution 1. Theorem: Suppose that f and $\partial_y f$ are continuous in a neighborhood of the point (x_0, y_0). Then the IVP $y' = f(x, y)$, $y'(x_0) = y_0$ has a unique solution in some neighborhood of x_0 (page 17 of the class notes).

For the given problem, $f(x, y) = (\sin y)^{-1}$ and $\partial_y f(x, y) = -(\sin y)^{-2} \cos y$, which are both continuous in the neighborhood of $(0, \pi/4)$, thus guaranteeing the existence of a unique solution near $x = 0$.

Question 2. (a) State a theorem for existence and uniqueness of solutions to the IVP

$$\begin{aligned} ay'' + by' + cy &= 0, \\ y(x_0) &= y_0, \\ y'(x_0) &= y_1, \end{aligned}$$

where a, b, and c are constants and $a \neq 0$.

(b) State a theorem for the existence and uniqueness of solutions to the IVP

$$\begin{aligned} x' &= Ax + f, \\ x(t_0) &= x_0, \end{aligned}$$

where A and f are, respectively, a matrix function and a vector function.

(c) Solve the IVP

$$\begin{aligned} 17y'' + \sqrt{3}y' + \frac{1}{19}y &= 0, \\ y(1) &= 0, \\ y'(1) &= 0, \end{aligned}$$

Hint: think carefully about this problem before doing any computation.

Solution 2. (a) Theorem: For any real numbers a, b, c, x_0, y_0, and y_1, $a \neq 0$, there exists a unique solution to the IVP

$$\begin{aligned} ay'' + by' + cy &= 0, \\ y(x_0) &= y_0, \\ y'(x_0) &= y_1. \end{aligned}$$

The solution is valid for all $t \in (-\infty, \infty)$ (page 47 of the class notes).

(b) Theorem: Let $A(t)$ and $f(t)$ be continuous on the interval I that contains t_0, where A is $n \times n$. Then, for any x_0, there exists a unique solution $x(t)$, defined on the whole interval I, to the IVP $x' = Ax + f$, $x(t_0) = x_0$ (page 116 of the class notes).

(c) The zero function satisfies the IVP. By the uniqueness stated in (a), it is the only solution.

Question 3. (a) State the definition of the exponential of a matrix and explain why it is well-defined.

(b) Show that e^{At} is a fundamental matrix for the system $x' = Ax$, where A is a constant $n \times n$ matrix.

Solution 3. If M is a $n \times n$ matrix, e^M is defined as

$$e^M = \sum_{j=0}^{\infty} \frac{M^j}{j!},$$

where $M^0 = I$. Introducing $\|M\|$ by

$$\|M\| = \sup_{\|x\|=1} \|Mx\|,$$

where the norms on the right hand side are the usual Euclidean norm, we can verify that this defines a norm on the space of $n \times n$ matrices and that for any two $n \times n$ matrices, $\|MN\| \leq \|M\| \|N\|$. Then

$$\left\|e^M\right\| \leq \sum_{j=0}^{\infty} \frac{1}{j!} \|M\|^j = e^{\|M\|} < \infty.$$

This shows absolute convergence of the series defining e^M, which implies convergence. (See pages 139-140 of the class notes.)

(b) We have

$$\frac{d}{dt} e^{At} = \frac{d}{dt} \sum_{j=0}^{\infty} \frac{A^j t^j}{j!} = \sum_{j=0}^{\infty} j \frac{A^j t^{j-1}}{j!} = A \sum_{j=0}^{\infty} \frac{A^j t^j}{j!} = A e^{At},$$

i.e., e^{At} satisfies the matrix differential equatoin $X' = AX$. Since e^{At} is invertible (recall that $(e^{At})^{-1} = e^{-At}$), its columns are linearly independent, so it is a fundamental matrix. (See pages 140-141 of the class notes.)

Question 4. The questions that follow refer to the system

$$\dot{x} = f(x, y),$$
$$\dot{y} = g(x, y).$$

(a) What is a critical point for this system and how is it related to solutions of the system?

(b) What is an isolated critical point?

(c) Define what it means to say that a critical point is stable, asymptotically stable, and unstable. Illustrate the definitions with pictures.

(d) Define an almost linear system near the origin.

Solution 4. (a) A point (x_0, y_0) is a critical point if it satisfies $f(x_0, y_0) = 0 = g(x_0, y_0)$. In this case, the constant functions $x(t) = x_0$, $y(t) = y_0$ are a solution to the system. (Page 149 of the class notes.)

(b) A critical point (x_0, y_0) is isolated if there exists a neighborhood D of (x_0, y_0) such that (x_0, y_0) is the only critical point in D (page 183 of the class notes).

(c) A critical point (x_0, y_0) is stable if, given $\varepsilon > 0$, there exists a $\delta > 0$ such that every solution $x(t)$, $y(t)$ satisfying

$$\sqrt{(x(0) - x_0)^2 + (y(0) - y_0)^2} < \delta$$

also satisfies

$$\sqrt{(x(t) - x_0)^2 + (y(t) - y_0)^2} < \varepsilon$$

for all $t \geq 0$. If (x_0, y_0) is stable and there exists a $\eta > 0$ such that any solution $x(t)$, $y(t)$ satisfying

$$\sqrt{(x(0) - x_0)^2 + (y(0) - y_0)^2} < \eta$$

converges to (x_0, y_0) as $t \to \infty$, then the critical point is called asymptotically stable. If a critical point is not stable then it is called unstable. (Page 167 of the class notes.) See page 168 of the class notes for illustrations.

(d) See page 171 of the class notes.

Question 5. Decide whether the systems below are almost linear near the origin. When they are, analyze the stability of the critical points.

(a)

$$\dot{x} = x + y + x^2 + y^2,$$
$$\dot{y} = 3x + 3y + x^2 y.$$

(b)

$$\dot{x} = -x + x^2 + y^2,$$
$$\dot{y} = x + 2y + xy\sin(xy).$$

Solution 5. (a) The functions x^2+y^2 and x^2y satisfy $(x^2+y^2)/\sqrt{x^2+y^2} \to 0$ and $(x^2y)/\sqrt{x^2+y^2} \to 0$ as $\sqrt{x^2+y^2} \to 0$. But the determinant of the corresponding linear part of the system is zero, thus this is not an almost linear system.

(b) Note that $(0,0)$ is a critical point. We have $(x^2+y^2)/\sqrt{x^2+y^2} \to 0$ as $\sqrt{x^2+y^2} \to 0$, and $|xy\sin(xy)| \leq |xy| \leq \frac{1}{2}(x^2+y^2)$ so that $(xy\sin(xy))/\sqrt{x^2+y^2} \to 0$ as $\sqrt{x^2+y^2} \to 0$. The determinant of the corresponding linear part is non-zero, hence this is an almost linear system. The eigenvalues of the linear part are -1 and 2, thus the origin is an unstable saddle point.

Question 6. Consider the DE

$$\ddot{x} + e^x - 1 = 0.$$

(a) Explain why this is a conservative system.

(b) Find the potential function G.

(c) Find the energy function $E(x, v)$. Select it so that $E(0, 0) = 0$.

(d) Write the DE as a first order system and determine its critical points.

(e) Determine the stability of the critical points.

Solution 6. (a) The system can be written as $\ddot{x} = F(x)$, thus this is a conservative system (see page 176 of the class notes).

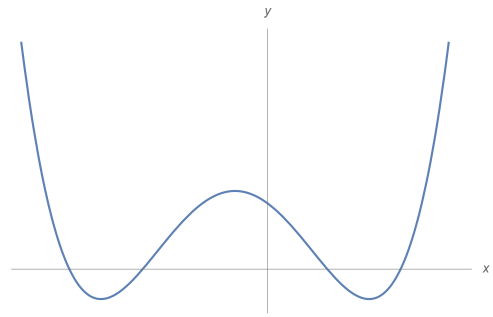
(b) We write $\ddot{x} + g(x) = 0$, with $g(x) = e^x - 1$. Then $G(x) = \int g(x)\, dx = e^x - x + C$ (see page 177 of the class notes).

(c) The energy function is given by $E(x, v) = \frac{1}{2}v^2 + G(x)$. Choosing $C = -1$ we have $E(x, v) = \frac{1}{2}v^2 + e^x - x - 1$, so $E(0, 0) = 0$. (Page 177 of the class notes).

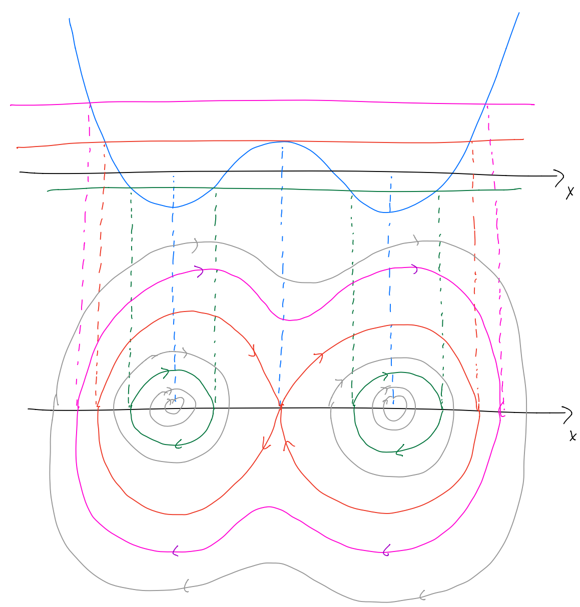
(d) We have $\dot{x} = v$, $\dot{v} = -e^x + 1$ (page 177 of the class notes).

(e) The only critical point is $(0, 0)$. Since $G'(0) = g(0) = 0$ and $G''(0) = 1 > 0$, $x = 0$ is a local minimum of G and we conclude that the critical point is a center (see pages 179-182 of the class notes).

Question 7. Consider a conservative system whose potential function is given by the graph below. Sketch the phase portrait of the system.



Solution 7. See pages 179-182 of the class notes for an explanation of how to draw the phase portrait in this situation. The picture is



Question 8. Consider the system

$$\dot{x} = x^3y + x^2y^3 - x^5,$$
$$\dot{y} = -2x^4 - 6x^3y^2 - 2y^5.$$

(a) Show that the origin is a critical point.

(b) Explain why this system is not almost linear.

(c) Determine the stability of the origin. To do so, you can use, without proving it, that the origin is an isolated critical point. *Hint:* the function $ax^2 + by^2$ is useful.

Solution 8. (a) This follows by plugging $(0, 0)$ into the right hand side and observing that the resulting expressions vanish.

(b) The corresponding linear part has $a = b = c = d = 0$ thus it fails the condition $ad - bc \neq 0$.

(c) Let $V(x, y) = ax^2 + by^2$ and compute

$$\begin{aligned}\frac{d}{dt}V(x, y) &= V_x(x, y)\dot{x} + V_y(x, y)\dot{y} \\ &= 2ax(x^3y + x^2y^3 - x^5) + 2by(-2x^4 - 6x^3y^2 - 2y^5) \\ &= 2a(x^4y + x^3y^3 - x^6) + 2b(-2x^4y - 6x^3y^3 - 2y^6).\end{aligned}$$

If we choose $a = 2$, $b = 1$, we get

$$\frac{d}{dt}V(x, y) = -8x^3y^3 - 4x^6 - 4y^6 = -4(x^3 + y^3)^2,$$

which is negative semi-definite. By the Lyapunov stability theorem (page 184 of the class notes), the critical point is stable.

Question 9. Prove that the equation

$$\ddot{x} + (x^4 + (\dot{x})^2 - 1)\dot{x} + x = 0$$

has a non-constant periodic solution.

Solution 9. Write the system as

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -(x^4 + y^2 - 1)y - x.\end{aligned}$$

We see that $(0, 0)$ is the only critical point of this system. Consider $V(x, y) = ax^m + by^n$. Then

$$\begin{aligned}\frac{d}{dt}V(x, y) &= amx^{m-1}\dot{x} + bny^{n-1}\dot{y} \\ &= amx^{m-1}y - bny^{n-1}((x^4 + y^2 - 1)y + x).\end{aligned}$$

If we choose $m = n = 2$ and $a = b = 1$ we find

$$\frac{d}{dt}V(x, y) = 2xy - 2y((x^4 + y^2 - 1)y + x) = -2y^2(x^4 + y^2 - 1) = y^2(1 - (x^4 + y^2)).$$

Consider the curve γ given by $x^4 + y^2 = 1$. Then, $\frac{d}{dt}V(x(t), y(t))$ is ≥ 0 inside γ and ≤ 0 outside γ. The curve γ lies between the circle $x^2 + y^2 = 1$ and the square $\{(x, y) \in \mathbb{R}^2 \mid \max(|x|, |y|) = 1\}$, touching them at the points $(\pm 1, 0)$ and $(0, \pm 1)$. We can choose as the region R of the Poincaré-Bendixson theorem (page 191 of the class notes) any annulus $r_A \leq x^2 + y^2 \leq r_B$ with $0 < r_A < 1$ and $r_B > \sqrt{2}$. Applying this theorem then gives the result.

Question 10. Review the class notes, examples, practice tests, and tests solutions posted in the course webpage. Be prepared to state a theorem that you need to use and to state definitions. The final exam may contain a true or false question; be prepared to justify your answers.

Solution 10. Done!