VANDERBILT UNIVERSITY

MATH 2610 – ORDINARY DIFFERENTIAL EQUATIONS

Practice for the final test

The final exam will be cumulative with an emphasis on chapter 12. The test will cover all material, except for Cauchy-Euler equations in section 4.7, sections 1.3, 1.4, 5.2, 5.3, non-conservative systems in section 12.4, and section 12.7.

Question 1. Consider the IVP $y' = f(x, y)$, $y'(x_0) = y_0$. State a theorem that guarantees the existence of a unique solution defined in a neighborhood of x_0. Using the theorem that you stated, show that $y' = (\sin y)^{-1}$, $y(0) = \pi/4$ admits a unique solution in some neighborhood of 0.

Question 2. (a) State a theorem for existence and uniqueness of solutions to the IVP

$$\begin{aligned} ay'' + by' + cy &= 0, \\ y(x_0) &= y_0, \\ y'(x_0) &= y_1, \end{aligned}$$

where a, b, and c are constants and $a \neq 0$.

(b) State a theorem for the existence and uniqueness of solutions to the IVP

$$\begin{aligned} x' &= Ax + f, \\ x(t_0) &= x_0, \end{aligned}$$

where A and f are, respectively, a matrix function and a vector function.

(c) Solve the IVP

$$\begin{aligned} 17y'' + \sqrt{3}y' + \frac{1}{19}y &= 0, \\ y(1) &= 0, \\ y'(1) &= 0, \end{aligned}$$

Hint: think carefully about this problem before doing any computation.

Question 3. (a) State the definition of the exponential of a matrix and explain why it is well-defined.

(b) Show that e^{At} is a fundamental matrix for the system $x' = Ax$, where A is a constant $n \times n$ matrix.

Question 4. The questions that follow refer to the system

$$\begin{aligned} \dot{x} &= f(x, y), \\ \dot{y} &= g(x, y). \end{aligned}$$

(a) What is a critical point for this system and how is it related to solutions of the system?

(b) What is an isolated critical point?

(c) Define what it means to say that a critical point is stable, asymptotically stable, and unstable. Illustrate the definitions with pictures.

(d) Define an almost linear system near the origin.

Question 5. Decide whether the systems below are almost linear near the origin. When they are, analyze the stability of the critical points.

(a)

$$\dot{x} = x + y + x^2 + y^2,$$
$$\dot{y} = 3x + 3y + x^2 y.$$

(b)

$$\dot{x} = -x + x^2 + y^2,$$
$$\dot{y} = x + 2y + xy\sin(xy).$$

Question 6. Consider the DE

$$\ddot{x} + e^x - 1 = 0.$$

(a) Explain why this is a conservative system.

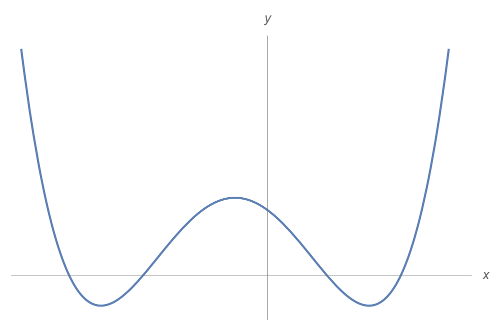
(b) Find the potential function G.

(c) Find the energy function $E(x, v)$. Select it so that $E(0, 0) = 0$.

(d) Write the DE as a first order system and determine its critical points.

(e) Determine the stability of the critical points.

Question 7. Consider a conservative system whose potential function is given by the graph below. Sketch the phase portrait of the system.



Question 8. Consider the system

$$\dot{x} = x^3 y + x^2 y^3 - x^5,$$
$$\dot{y} = -2x^4 - 6x^3 y^2 - 2y^5.$$

(a) Show that the origin is a critical point.

(b) Explain why this system is not almost linear.

(c) Determine the stability of the origin. To do so, you can use, without proving it, that the origin is an isolated critical point. *Hint:* the function $ax^2 + by^2$ is useful.

Question 9. Prove that the equation

$$\ddot{x} + (x^4 + (\dot{x})^2 - 1)\dot{x} + x = 0$$

has a non-constant periodic solution.

Question 10. Review the class notes, examples, practice tests, and tests solutions posted in the course webpage. Be prepared to state a theorem that you need to use and to state definitions. The final exam may contain a true or false question; be prepared to justify your answers.