VANDERBILT UNIVERSITY

MATH 2610 – ORDINARY DIFFERENTIAL EQUATIONS

Final exam – solutions

NAME: Solutions.

Directions. This exam contains nine questions. Make sure you clearly indicate the pages where your solutions are written. Answers without justification will receive little or no credit. Write clearly, legibly, and in a logical fashion. Make precise statements (for instance, write an equal sign if two expressions are equal; say that one expression is a consequence of another when this is the case, etc.).

If you need to use a theorem that was stated in class, you do not need to prove it, unless a question explicitly says so. You do need, however, to state the theorems you invoke.

If you do not understand a question, or think that some problem is ambiguous, missing information, or incorrectly stated, write how you interpret the problem and solve it accordingly.

Question	Points
1 (10 pts)	
2 (10 pts)	
3 (10 pts)	
4 (10 pts)	
5 (10 pts)	
6 (10 pts)	
7 (15 pts)	
8 (15 pts)	
9 (10 pts)	
TOTAL (100 pts)	

Question 1. (10 pts) (a) State a theorem that guarantees that the IVP

$$y' = f(x, y),$$
$$y(x_0) = y_0,$$

has a unique solution defined in some neighborhood of x_0.

(b) Using the theorem you stated in (a), show that the IVP

$$y' = (\cos y)^{-1},$$
$$y(0) = \frac{\pi}{6},$$

has a unique solution in some neighborhood of 0.

(c) State a theorem for the existence and uniqueness of solutions to the IVP

$$x' = Ax + f,$$
$$x(t_0) = x_0,$$

where A and f are, respectively, a matrix function and a vector function.

(d) Solve the IVP

$$x' = \begin{bmatrix} \sqrt{3} & -119 \\ 7 & \sqrt{11} \end{bmatrix} x,$$
$$x(0) = (0, 0).$$

Hint: think carefully about this problem before doing any computation.

Solution 1. (a) Theorem: Suppose that f and $\partial_y f$ are continuous in a neighborhood of the point (x_0, y_0). Then the IVP $y' = f(x, y)$, $y'(x_0) = y_0$ has a unique solution in some neighborhood of x_0.

(b) For the given problem, $f(x, y) = (\cos y)^{-1}$ and $\partial_y f(x, y) = (\cos y)^{-2} \sin y$, which are both continuous in the neighborhood of $(0, \pi/4)$, thus guaranteeing the existence of a unique solution near $x = 0$ by the theorem stated in (a).

(c) Theorem: Let $A(t)$ and $f(t)$ be continuous on the interval I that contains t_0, where A is $n \times n$. Then, for any x_0, there exists a unique solution $x(t)$, defined on the whole interval I, to the IVP $x' = Ax + f$, $x(t_0) = x_0$.

(d) $x(t) = 0$ solves the IVP; by the theorem stated in (c) this is the only solution.

Question 2. (10 pts) (a) State the definition of the exponential of a matrix and explain why it is well-defined.

(b) Show that e^{At} is a fundamental matrix for the system $x' = Ax$, where A is a constant $n \times n$ matrix.

Solution 2. If M is a $n \times n$ matrix, e^M is defined as

$$e^M = \sum_{j=0}^{\infty} \frac{M^j}{j!},$$

where $M^0 = I$. Introducing $\|M\|$ by

$$\|M\| = \sup_{\|x\|=1} \|Mx\|,$$

where the norms on the right hand side are the usual Euclidean norm, we can verify that this defines a norm on the space of $n \times n$ matrices and that for any two $n \times n$ matrices, $\|MN\| \leq \|M\| \|N\|$. Then

$$\left\|e^M\right\| \leq \sum_{j=0}^{\infty} \frac{1}{j!} \|M\|^j = e^{\|M\|} < \infty.$$

This shows absolute convergence of the series defining e^M, which implies convergence.

(b) We have

$$\frac{d}{dt} e^{At} = \frac{d}{dt} \sum_{j=0}^{\infty} \frac{A^j t^j}{j!} = \sum_{j=0}^{\infty} j \frac{A^j t^{j-1}}{j!} = A \sum_{j=0}^{\infty} \frac{A^j t^j}{j!} = Ae^{At},$$

i.e., e^{At} satisfies the matrix differential equatoin $X' = AX$. Since e^{At} is invertible (recall that $(e^{At})^{-1} = e^{-At}$), its columns are linearly independent, so it is a fundamental matrix.

Question 3. (10 pts) Decide whether the systems below are almost linear near the origin. When they are, analyze the stability of $(0,0)$.

(a)

$$\dot{x} = \sin(y - 3x),$$
$$\dot{y} = \cos x - e^y.$$

(b)

$$\dot{x} = x - y - x^4,$$
$$\dot{y} = -3x + 3y + y^4.$$

Solution 3. (a) We need first to rewrite the system in the form

$$\dot{x} = ax + by + f(x,y)$$
$$\dot{y} = cx + dy + g(x,y).$$

Since sin, cos, and the exponential have Taylor expansions that converge for all values, we can write

$$\sin(y - 3x) = y - 3x - \frac{(y-3x)^3}{3!} + O((y-3x)^5),$$

where $O(z^m)$ means terms involving powers of z of order at least m. Because we are interested in the limit $\sqrt{x^2+y^2} \to 0$, we can assume that $|y-3x| < 1$, $|x| < 1$, and $|y| < 1$, in which case

$$|\frac{(y-3x)^3}{3!} + O((y-3x)^5)| \le O(|y-3x|^3) \le O(x^2+y^2),$$

implying

$$\lim_{\sqrt{x^2+y^2}\to 0} \frac{-\frac{(y-3x)^3}{3!} + O((y-3x)^5)}{\sqrt{x^2+y^2}} = 0.$$

Similarly,

$$\cos x - e^y = 1 - \frac{x^2}{2!} + O(x^4) - (1 + y + O(y^2)) = -y + O(x^2+y^2)$$

and $O(x^2+y^2)/\sqrt{x^2+y^2} \to 0$ as $\sqrt{x^2+y^2} \to 0$.

The system now has $a = -3$, $b = 1$, $c = 0$, $d = -1$, and is almost linear. The eigenvalues of the associated linear system are -3 and -1, and we conclude that $(0,0)$ is an asymptotically stable improper node.

(b) The system fails the condition $ad - bc \neq 0$, thus it is not almost linear.

Question 4. (10 pts) Consider the DE

$$\ddot{x} + x^2 - 1 = 0.$$

(a) Explain why this is a conservative system.

(b) Find the potential function G.

(c) Find the energy function $E(x, v)$. Select it so that $E(0, 0) = 0$.

(d) Write the DE as a first order system and determine its critical points.

(e) Determine the stability of the critical points.

Solution 4. (a) It can be written in the form $\ddot{x} = F(x)$, thus it is a conservative system.

(b) Writing $\ddot{x}+g(x) = 0$, we have $g(x) = x^2-1$. Then G is given by $G(x) = \int g(x)\,dx = \frac{1}{3}x^3-x+C$.

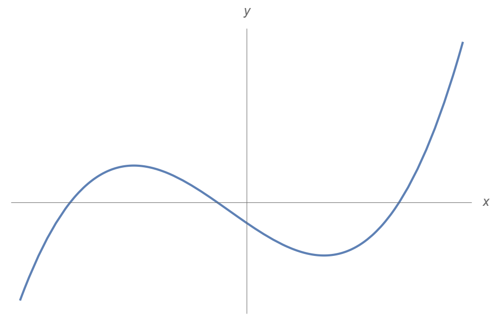
(c) We have $E(x, v) = \frac{1}{2}v^2 + G(x)$. For $E(0, 0) = 0$, we choose $C = 0$, so $E(x, v) = \frac{1}{2}v^2 + \frac{1}{3}x^3 - x$.

(d) Setting $\dot{x} = v$, we have

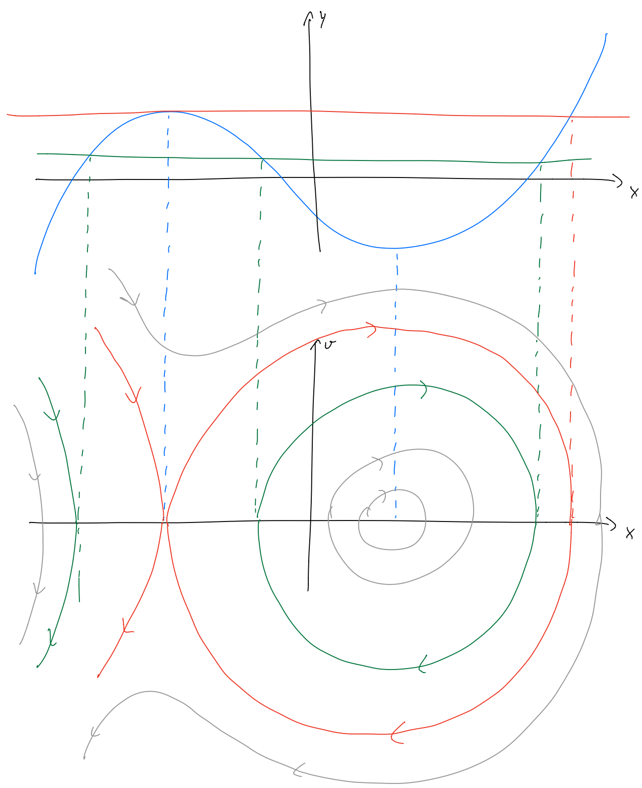
$$\dot{x} = v,$$
$$\dot{v} = -x^2 + 1.$$

(e) We first find the x's such that $G'(x) = g(x) = x^2 - 1 = 0$, giving ± 1. Thus $(-1, 0)$ and $(1, 0)$ are the critical points. Next, we compute $G''(-1) = -2 < 0$ and $G''(1) = 2 > 0$, thus -1 is a local maximum and 1 a local minimum of G. We conclude that $(-1, 0)$ is a saddle point, thus unstable, and that $(1, 0)$ is a center, thus stable.

Question 5. (10 pts) Consider a conservative system whose potential function G is given by the graph below. Sketch the phase portrait of the system.



Solution 5.



Question 6. (10 pts) Consider the system

$$\begin{aligned}\dot{x} &= y^3 - 2x^3,\\ \dot{y} &= -3x - y^3.\end{aligned}$$

(a) Show that the origin is a critical point.

(b) Determine the stability of the origin. To do so, you can use, without proving it, that the origin is an isolated critical point. *Hint:* the function $ax^2 + by^4$ is useful.

Solution 6. (a) Plugging $x = 0 = y$, the right hand side of the system vanishes, thus $(0, 0)$ is a critical point.

(b) Set $V(x, y) = ax^2 + by^4$ and compute

$$\begin{aligned}\frac{d}{dt}V(x, y) &= V_x(x, y)\dot{x} + V_y(x, y)\dot{y} = 2ax(y^3 - 2x^3) + 4by^3(-3x - y^3)\\ &= (2a - 12b)xy^3 - 4ax^4 - 4by^6.\end{aligned}$$

Choosing $b = 1$ and $a = 6$, we find

$$\frac{d}{dt}V(x, y) = -24x^4 - 4y^6,$$

which is negative definite. Since V is positive definite with this choice of a and b, we have verified the hypotheses of Lyapunov's stability theorem, and conclude that the origin is asymptotically stable.

Question 7. (15 pts) Consider the system

$$\dot{x} = \frac{1}{4}x^4 y,$$
$$\dot{y} = \frac{1}{2}x^3 y^2.$$

Determine the regions in the xy-plane where the system does not admit any non-constant periodic solutions. Said differently, find all regions D in $\mathbb{R}^2$ with the property that if $(x(t), y(t))$ is a periodic solution that lies entirely in D then $(x(t), y(t))$ is a constant solution.

Solution 7. We will use Bendixson negative criterion. Letting $f(x, y) = \frac{1}{4}x^4 y$ and $g(x, y) = \frac{1}{2}x^3 y^2$, we find

$$f_x(x, y) + g_y(x, y) = x^3 y + x^3 y = 2x^3 y.$$

Since x^3 has the same sign as x, $f_x(x, y) + g_y(x, y)$ does not change sign on each quadrant on the plane. Therefore, by Bendixson's criterion, the system does not have any closed non-trivial trajectory contained in the first, second, third, or fourth quadrant.

Question 8. (15 pts) Prove that the system

$$\dot{x} = y^3 - 3x^3 - xy^2 + 6x,$$
$$\dot{y} = -x.$$

has a non-constant periodic solution. *Hint:* the function $ax^2 + by^4$ is useful.

Solution 8. Note that $(0,0)$ is the only critical point of the system. Set $V(x,y) = ax^2 + by^4$ and compute

$$\begin{aligned}\frac{d}{dt}V(x,y) &= V_x(x,y)\dot{x} + V_y(x,y)\dot{y} = 2ax(y^3 - 3x^3 - xy^2 + 6x) + 4by^3(-x) \\ &= (2a-4b)xy^3 - 6ax^4 - 2ax^2y^2 + 12ax^2.\end{aligned}$$

If we choose $a = 2$, $b = 1$,

$$\frac{d}{dt}V(x,y) = -4x^2(3x^2 + y^2 - 6).$$

Letting γ be the ellipse $3x^2 + y^2 = 6$, we have that $\frac{d}{dt}V(x(t), y(t)) \leq 0$ for $(x(t), y(t))$ outside γ and $\frac{d}{dt}V(x(t), y(t)) \geq 0$ for $(x(t), y(t))$ inside γ. Because γ does not contain the origin, we can find two closed curves C_A and C_B containing the origin and not intersecting γ, such that C_A is contained in the region bounded by γ and γ is contained in the region bounded by C_B. It follows that if we let R be the closed region between C_A and C_B, then trajectories starting in R stay in R for all $t \geq 0$. We have thus verified the hypotheses of the Poincaré-Bendixson theorem. We conclude that there exists a non-trivial periodic solution to the system.

Question 9. (10 pts) True or false? Justify your answers.

(a) Let A be a $n \times n$ matrix and u a generalized eigenvector of A associated to an eigenvalue λ. Then $e^{\lambda t}u$ is a solution to $x' = Ax$.

(b) Let $x_1, \cdots, x_n$ be n-component vector functions defined on an interval I. Assume that $W[x_1, \ldots, x_n](t) = 0$ for all $t \in I$. Then $x_1, \ldots, x_n$ are linearly dependent.

(c) If the system

$$\dot{x} = f(x, y),$$
$$\dot{y} = g(x, y),$$

has a limit cycle, then it must have at least one constant solution.

(d) Suppose a 2×2 system of DE has the property that any solution that starts in the region R given by

$$R = \{(x, y) \in R^2 \,|\, x^2 + y^2 \geq 1\}$$

stays in the region R. Suppose further that $(0, 0)$ is the only critical point of the system. Then, by the Poincaré-Bendixson theorem, this system has at least on non-constant periodic solution.

Solution 9. (a) False. We have $(e^{\lambda t}u)' = \lambda e^{\lambda t}u$. If this equals $Ae^{\lambda t}u$, then $\lambda e^{\lambda t}u = Ae^{\lambda t}u$, or $(A - \lambda I)u = 0$. This means that u is an eigenvector of A. If u is a generalized eigenvector but not an eigenvector, then the equality does not hold.

(b) False. For example, $(t, |t|)$ and $(|t|, t)$ are linearly independent on $(-\infty, \infty)$, but their Wronskian is zero.

(c) True. A limit cycle always contains a critical point, thus a constant solution.

(d) False. The given region R is not bounded, thus the Poincaré-Bendixson theorem does no apply.