VANDERBILT UNIVERSITY

MATH 2610 – ORDINARY DIFFERENTIAL EQUATIONS

Linear algebra examples

Question 1. Use Gauss-Jordan elimination to solve the system:

$$\begin{cases} x + 3y + 2z = 2 \\ 2x + 7y + 7z = -1 \\ 2x + 5y + 2z = 7 \end{cases}$$

Question 2. Compute AB if

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 4 \\ 2 & -3 & 5 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 7 & -4 & 3 \\ 1 & 5 & -2 \\ 0 & 3 & 9 \end{bmatrix}.$$

Question 3. Find the inverse of

$$A = \begin{bmatrix} 3 & 5 & 6 \\ 2 & 4 & 3 \\ 2 & 3 & 5 \end{bmatrix}$$

Question 4. Find the determinant of

$$A = \begin{bmatrix} 3 & 5 & 6 \\ 2 & 4 & 3 \\ 2 & 3 & 5 \end{bmatrix}$$

Question 5. Solve the system

$$A\vec{x} = \vec{b},$$

where A is the matrix of question 3 and

$$\vec{b} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix},$$

by using the inverse of A.

Question 6. Solve the system of question 5 by using Cramer's rule.

SOLUTIONS.

Question 1. The augmented matrix of the system is

$$\begin{bmatrix} 1 & 3 & 2 & \vdots & 2 \\ 2 & 7 & 7 & \vdots & -1 \\ 2 & 5 & 2 & \vdots & 7 \end{bmatrix}$$

Then

$$\begin{bmatrix} 1 & 3 & 2 & \vdots & 2 \\ 2 & 7 & 7 & \vdots & -1 \\ 2 & 5 & 2 & \vdots & 7 \end{bmatrix} \underset{\sim}{\substack{L_2 \leftarrow -2L_1 + L_2 \\ L_3 \leftarrow -2L_1 + L_3}} \begin{bmatrix} 1 & 3 & 2 & \vdots & 2 \\ 0 & 1 & 3 & \vdots & -5 \\ 0 & -1 & -2 & \vdots & 3 \end{bmatrix}$$

$$\underset{\sim}{L_3 \leftarrow L_2 + L_3} \begin{bmatrix} 1 & 3 & 2 & \vdots & 2 \\ 0 & 1 & 3 & \vdots & -5 \\ 0 & 0 & 1 & \vdots & -2 \end{bmatrix} \underset{\sim}{\substack{L_2 \leftarrow -3L_3 + L_2 \\ L_1 \leftarrow -2L_3 + L_1}} \begin{bmatrix} 1 & 3 & 0 & \vdots & 6 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & -2 \end{bmatrix}$$

$$\underset{\sim}{L_1 \leftarrow -3L_2 + L_1} \begin{bmatrix} 1 & 0 & 0 & \vdots & 3 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & -2 \end{bmatrix}$$

Therefore the solution of the system is $x = 3$, $y = 1$, $z = -2$.

Question 2. Let us compute the product of A with each column of B.

$$\begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 4 \\ 2 & -3 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ 1 \\ 0 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} + 0 \begin{bmatrix} -3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 23 \\ 11 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 4 \\ 2 & -3 & 5 \end{bmatrix} \begin{bmatrix} -4 \\ 5 \\ 3 \end{bmatrix} = -4 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} + 3 \begin{bmatrix} -3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -13 \\ 10 \\ -8 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 4 \\ 2 & -3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 9 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} + 9 \begin{bmatrix} -3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -24 \\ 41 \\ 57 \end{bmatrix}.$$

Therefore

$$AB = \begin{bmatrix} 7 & -13 & -24 \\ 23 & 10 & 41 \\ 11 & -8 & 57 \end{bmatrix}.$$

Question 3. Write

$$\left[\begin{array}{ccc|ccc} 3 & 5 & 6 & 1 & 0 & 0 \\ 2 & 4 & 3 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right]$$

and apply Gauss-Jordan elimination.

$$\left[\begin{array}{ccc|ccc} 3 & 5 & 6 & 1 & 0 & 0 \\ 2 & 4 & 3 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{L_1 \leftarrow L_1 - L_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & -1 & 0 \\ 2 & 4 & 3 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{L_2 \leftarrow L_2 - L_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 & 1 & -1 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{L_3 \leftarrow L_3 - 2L_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 & 1 & -1 \\ 0 & 1 & -1 & -2 & 2 & 1 \end{array}\right]$$

$$\underset{\sim}{L_3 \leftarrow L_3 - L_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right] \underset{\sim}{\begin{array}{c} L_2 \leftarrow 2L_3 + L_2 \\ L_2 \leftarrow -3L_3 + L_1 \end{array}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 7 & -4 & -6 \\ 0 & 1 & 0 & -4 & 3 & 3 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right] \underset{\sim}{L_1 \leftarrow L_1 - L_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 11 & -7 & -9 \\ 0 & 1 & 0 & -4 & 3 & 3 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right]$$

So

$$A^{-1} = \begin{bmatrix} 11 & -7 & -9 \\ -4 & 3 & 3 \\ -2 & 1 & 2 \end{bmatrix}.$$

Question 4. We have

$$A_{11} = \begin{bmatrix} 4 & 3 \\ 3 & 5 \end{bmatrix} \Rightarrow \det(A_{11}) = 4 \cdot 5 - 3 \cdot 3 = 11,$$

$$A_{12} = \begin{bmatrix} 2 & 3 \\ 2 & 5 \end{bmatrix} \Rightarrow \det(A_{12}) = 2 \cdot 5 - 3 \cdot 2 = 4,$$

$$A_{13} = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} \Rightarrow \det(A_{13}) = 2 \cdot 3 - 2 \cdot 4 = -2.$$

Hence,

$$\det(A) = 3\det(A_{11}) - 5\det(A_{12}) + 6\det(A_{13}) = 33 - 20 - 12 = 1.$$

Question 5. Write

$$\left[\begin{array}{ccc|ccc} 3 & 5 & 6 & 1 & 0 & 0 \\ 2 & 4 & 3 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right]$$

and apply Gauss-Jordan elimination.

$$\left[\begin{array}{ccc|ccc} 3 & 5 & 6 & 1 & 0 & 0 \\ 2 & 4 & 3 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{L_1 \leftarrow L_1 - L_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & -1 & 0 \\ 2 & 4 & 3 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{L_2 \leftarrow L_2 - L_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 & 1 & -1 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{L_3 \leftarrow L_3 - 2L_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 & 1 & -1 \\ 0 & 1 & -1 & -2 & 2 & 1 \end{array}\right]$$

$$\underset{\sim}{L_3 \leftarrow L_3 - L_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right] \begin{array}{c} L_2 \leftarrow 2L_3 + L_2 \\ \underset{\sim}{L_2 \leftarrow -3L_3 + L_1} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 7 & -4 & -6 \\ 0 & 1 & 0 & -4 & 3 & 3 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right] \underset{\sim}{L_1 \leftarrow L_1 - L_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 11 & -7 & -9 \\ 0 & 1 & 0 & -4 & 3 & 3 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right]$$

So

$$A^{-1} = \begin{bmatrix} 11 & -7 & -9 \\ -4 & 3 & 3 \\ -2 & 1 & 2 \end{bmatrix}.$$

Therefore

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 11 & -7 & -9 \\ -4 & 3 & 3 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -14 \\ 6 \\ 2 \end{bmatrix}.$$

Question 6. Replace the first column of A with $\vec{b}$ to get:

$$A_1(\vec{b}) = \begin{bmatrix} 0 & 5 & 6 \\ 2 & 4 & 3 \\ 0 & 3 & 5 \end{bmatrix}.$$

Analogously, replacing the second and third column produces

$$A_2(\vec{b}) = \begin{bmatrix} 3 & 0 & 6 \\ 2 & 2 & 3 \\ 2 & 0 & 5 \end{bmatrix},$$

and

$$A_3(\vec{b}) = \begin{bmatrix} 3 & 5 & 0 \\ 2 & 4 & 2 \\ 2 & 3 & 0 \end{bmatrix}$$

Then

$$\det(A_1(\vec{b})) = 0 \cdot \det \begin{bmatrix} 4 & 2 \\ 3 & 0 \end{bmatrix} - 2 \cdot \det \begin{bmatrix} 5 & 6 \\ 3 & 5 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} 5 & 6 \\ 4 & 3 \end{bmatrix} = -14.$$

Analogously,

$$\det(A_2(\vec{b})) = -0 \cdot \det \begin{bmatrix} 2 & 3 \\ 2 & 50 \end{bmatrix} + 2 \cdot \det \begin{bmatrix} 3 & 6 \\ 2 & 5 \end{bmatrix} - 0 \cdot \det \begin{bmatrix} 3 & 6 \\ 2 & 3 \end{bmatrix} = 6,$$

$$\det(A_3(\vec{b})) = 0 \cdot \det \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} - 2 \cdot \det \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} = 2.$$

We finally obtain

$$x_1 = \frac{\det(A_1(\vec{b}))}{\det(A)} = \frac{-14}{1} = -14,$$

$$x_2 = \frac{\det(A_2(\vec{b}))}{\det(A)} = \frac{6}{1} = 6,$$

$$x_3 = \frac{\det(A_3(\vec{b}))}{\det(A)} = \frac{2}{1} = 2,$$

in accordance to what we found in question 5.