VANDERBILT UNIVERSITY

MATH 2610 – ORDINARY DIFFERENTIAL EQUATIONS

Examples of section 9.8

Question 1. Determine e^{At} by using generalized eigenvectors to find a fundamental matrix if

$$A = \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix}.$$

Solutions.

1. A simple computation gives

$$\det \begin{bmatrix} 5-\lambda & -4 & 0 \\ 1 & -\lambda & 2 \\ 0 & 2 & 5-\lambda \end{bmatrix} = -\lambda(\lambda-5)^2,$$

so $\lambda_1 = 0$ and $\lambda_2 = 5$ are the eigenvalues, with λ_2 of multiplicity two.

To find an eigenvector associated with λ_1, we solve

$$\left[\begin{array}{ccc|c} 5 & -4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 5 & 0 \end{array}\right].$$

Applying Gauss-Jordan elimination we find $u_1 = (-4, -5, 2)$, and $x_1 = e^{0t}u_1 = (-4, -5, 2)$ is a solution to $x' = Ax$.

Next, we move to λ_2, and consider:

$$\left[\begin{array}{ccc|c} 0 & -4 & 0 & 0 \\ 1 & -5 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right].$$

Applying Gauss-Jordan elimination, we find

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, this system has only one free variable, yielding only one linearly independent eigenvector which we can take to be $u_2 = (-2, 0, 1)$. Hence $x_2 = e^{5t}(-2, 0, 1)$ is a second linearly independent solution to $x' = Ax$. To find a third linearly independent solution, we need to find a generalized eigenvector associated with $\lambda_2 = 5$. Compute

$$(A - 5I)^2 = \begin{bmatrix} 0 & -4 & 0 \\ 1 & -5 & 2 \\ 0 & 2 & 0 \end{bmatrix}^2 = \begin{bmatrix} -4 & 20 & -8 \\ -5 & 25 & -10 \\ 2 & -10 & 4 \end{bmatrix}.$$

Now we solve

$$\left[\begin{array}{rrr|r} -4 & 20 & -8 & 0 \\ -5 & 25 & -10 & 0 \\ 2 & -10 & 4 & 0 \end{array}\right].$$

Applying Gauss-Jordan elimination gives

$$\left[\begin{array}{rrr|r} -1 & 5 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which has two free variables that yield two linearly independent generalized eigenvectors $u_2 = (-2, 0, 1)$ and $u_3 = (5, 1, 0)$ (notice that we already knew from above that u_2 is a solution since it is an eigenvector). To find a third (linearly independent) solution to $x' = Ax$, compute

$$x_3 = e^{At}u_3 = e^{5t}(u_3 + t(A - 5I)u_3) = e^{5t}\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + te^{5t}\begin{bmatrix} 0 & -4 & 0 \\ 1 & -5 & 2 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = e^{5t}\begin{bmatrix} 5 - 4t \\ 1 \\ 2t \end{bmatrix}.$$

A fundamental matrix is now given by $X = [x_1\, x_2\, x_3]$, i.e.,

$$X(t) = \begin{bmatrix} -4 & -2e^{5t} & e^{5t}(5 - 4t) \\ -5 & 0 & e^{5t} \\ 2 & e^{5t} & 2e^{5t}t \end{bmatrix}.$$

Recall that $e^{At} = X(t)(X(0))^{-1}$. Plugging $t = 0$ into $X(t)$ and using Gauss-Jordan elimination we find

$$(X(0))^{-1} = \frac{1}{25}\begin{bmatrix} 1 & -5 & 2 \\ -2 & 10 & 21 \\ 5 & 0 & 10 \end{bmatrix}.$$

Thus,

$$\begin{aligned} e^{At} = X(t)(X(0))^{-1} &= \frac{1}{25}\begin{bmatrix} -4 & -2e^{5t} & e^{5t}(5 - 4t) \\ -5 & 0 & e^{5t} \\ 2 & e^{5t} & 2e^{5t}t \end{bmatrix}\begin{bmatrix} 1 & -5 & 2 \\ -2 & 10 & 21 \\ 5 & 0 & 10 \end{bmatrix} \\ &= \frac{1}{25}\begin{bmatrix} -4 + 29e^{5t} - 20te^{5t} & 20 - 20e^{5t} & -8 + 8e^{5t} - 40te^{5t} \\ -5 + 5e^{5t} & 25 & -10 + 10e^{5t} \\ 2 - 2e^{5t} + 10te^{5t} & -10 + 10e^{5t} & 4 + 21e^{5t} + 20te^{5t} \end{bmatrix}. \end{aligned}$$