VANDERBILT UNIVERSITY

MATH 2610 – ORDINARY DIFFERENTIAL EQUATIONS

Examples of section 9.7

Question 1. Find the general solution of

$$\vec{x}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 6e^{3t} \\ 2e^{3t} \end{bmatrix}.$$

Question 2. Find the general solution of the system of question 1 using variation of parameters.

Solutions.

1. First, we compute the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. They are $\lambda_1 = 3$ and $\lambda_2 = -1$. Two eigenvectors associated with λ_1 and λ_2 are, respectively,

$$u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } u_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

so that

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} \text{ and } x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$

are two linearly independent solutions of the associated homogeneous equation.

As the inhomogeneous term in the equation is of the form (vector)$\times e^{3t}$, in order to find a particular solution, we try

$$x_p = ae^{3t} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{3t}.$$

Plugging into the equation yields

$$3 \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{3t} = \begin{bmatrix} a_1 + 2a_2 \\ 2a_1 + a_2 \end{bmatrix} e^{3t} + \begin{bmatrix} 6 \\ 2 \end{bmatrix} e^{3t}.$$

This leads to

$$\begin{cases} 2a_1 - 2a_2 & = 6, \\ -2a_1 + 2a_2 & = 2, \end{cases}$$

which is an inconsistent system. Therefore, we change our initial guess and now attempt

$$x_p = ate^{3t} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} te^{3t}.$$

Plugging into the equation,

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{3t} = \begin{bmatrix} -2a_1 + 2a_2 \\ 2a_1 + 2a_2 \end{bmatrix} te^{3t} + \begin{bmatrix} 6 \\ 2 \end{bmatrix} e^{3t}.$$

Setting the terms with t and without t on each side equal to each other produces

$$\begin{cases} 2a_1 - 2a_2 & = 0, \\ -2a_1 + 2a_2 & = 0, \end{cases}$$

and

$$\begin{cases} a_1 &= 6, \\ a_2 &= 2. \end{cases}$$

It is impossible to satisfy both systems at the same time, thus, again, our attempt has failed to produce a particular solution.

Following the ideas developed in class, we now try

$$x_p = ate^{3t} + be^{3t} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} te^{3t} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} e^{3t}.$$

Plugging into the equation gives

$$\begin{bmatrix} 6 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} 2a_1 - 2a_2 \\ -2a_1 + 2a_2 \end{bmatrix} te^{3t} + \begin{bmatrix} a_1 + 2b_1 - 2b_2 \\ a_2 - 2b_1 + 2b_2 \end{bmatrix} e^{3t}.$$

Setting the terms with t and without t on each side equal to each other,

$$\begin{cases} 2a_1 - 2a_2 &= 0, \\ -2a_1 + 2a_2 &= 0, \end{cases}$$

and

$$\begin{cases} a_1 + 2b_1 - 2b_2 &= 6, \\ a_2 - 2b_1 + 2b_2 &= 2. \end{cases}$$

In other words, we obtain the following system of four unknowns and four equations:

$$\begin{cases} 2a_1 - 2a_2 &= 0, \\ -2a_1 + 2a_2 &= 0, \\ a_1 + 2b_1 - 2b_2 &= 6, \\ a_2 - 2b_1 + 2b_2 &= 2. \end{cases}$$

Using Gauss-Jordan elimination, we find $a_1 = 4$, $a_2 = 4$, $b_1 = 1 + b_2$, and b_2 undetermined (i.e., a free variable). As discussed in class, we can set $b_2 = 0$, finally obtaining

$$x_p = \begin{bmatrix} 4 \\ 4 \end{bmatrix} te^{3t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t}.$$

The general solution is then $x = c_1 x_1 + c_2 x_2 + x_p$, where c_1 and c_2 are arbitrary constants.

2. From the previous question, we have a fundamental matrix

$$X(t) = \begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix}.$$

Recalling that an invertible matrix of the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

has inverse given by

$$\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

one immediately finds

$$(X(t))^{-1} = \frac{1}{2} \begin{bmatrix} e^{-3t} & e^{-3t} \\ -e^{t} & e^{t} \end{bmatrix}.$$

Next, invoke the formula

$$x_p = X(t)\int (X(t))^{-1} f(t)\,dt = \frac{1}{2}\begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix} \int \begin{bmatrix} e^{-3t} & e^{-3t} \\ -e^{t} & e^{t} \end{bmatrix} \begin{bmatrix} 6 \\ 2 \end{bmatrix} e^{3t}\,dt,$$

which gives

$$x_p = \frac{1}{2}\begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix} \int \begin{bmatrix} 8 \\ -4e^{4t} \end{bmatrix} dt.$$

Performing the integral:

$$x_p = \begin{bmatrix} 4 \\ 4 \end{bmatrix} te^{3t} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t}.$$

To see that this agrees with the previous solution, write

$$\frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$$

and recall that $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$ is a solution of the associated homogeneous equation.