VANDERBILT UNIVERSITY

MATH 2610 – ORDINARY DIFFERENTIAL EQUATIONS

Examples of section 9.6

Question 1. Find the general solution of

$$\vec{x}' = \begin{bmatrix} -3 & -2 \\ 9 & 3 \end{bmatrix} \vec{x}.$$

Question 2. Find the general solution of

$$\vec{x}' = \begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix} \vec{x}.$$

Solutions.

1. The characteristic equation is

$$\det \begin{bmatrix} -3-\lambda & -2 \\ 9 & 3-\lambda \end{bmatrix} = -9 + \lambda^2 + 18 = \lambda^2 + 9 = 0,$$

whose solutions are

$$\lambda_1 = 3i,\ \lambda_2 = -3i.$$

Recall that we saw in class that in the complex root case, the first root already gives two linearly independent solutions, so it is enough to consider $\lambda_1 = 3i$. We want to solve

$$\begin{bmatrix} -3-3i & -2 \\ 9 & 3-3i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Using Gauss-Jordan elimination and ignoring the free variable we find

$$v = \begin{bmatrix} -2 \\ 3+3i \end{bmatrix}.$$

This gives

$$x = \begin{bmatrix} -2 \\ 3+3i \end{bmatrix} e^{3it}.$$

Next, we separate the real and imaginary parts,

$$\begin{aligned} x = \begin{bmatrix} -2e^{3it} \\ (3+3i)e^{3it} \end{bmatrix} &= \begin{bmatrix} -2\cos(3t) - 2i\sin(3t) \\ (3+3i)(\cos(3t) + i\sin(3t)) \end{bmatrix} \\ &= \begin{bmatrix} -2\cos(3t) - 2i\sin(3t) \\ 3\cos(3t) - \sin(3t) + i(3\cos(3t) + -3\sin(3t)) \end{bmatrix} \\ &= \begin{bmatrix} -2\cos(3t) \\ 3\cos(3t) - 3\sin(3t) \end{bmatrix} + i \begin{bmatrix} -2\sin(3t) \\ 3\sin(3t) + 3\cos(3t) \end{bmatrix}. \end{aligned}$$

Hence the two linearly independent solutions are

$$x_1 = \begin{bmatrix} -2\cos(3t) \\ 3\cos(3t) - 3\sin(3t) \end{bmatrix},$$

$$x_2 = \begin{bmatrix} -2\sin(3t) \\ 3\sin(3t) + 3\cos(3t) \end{bmatrix}.$$

2. As before, we look for solutions of the characteristic equation

$$\det \begin{bmatrix} 5-\lambda & 5 & 2 \\ -6 & -6-\lambda & -5 \\ 6 & 6 & 5-\lambda \end{bmatrix} = 0.$$

The solutions are

$$\lambda_1 = 0,\ \lambda_2 = 2 \pm 3i \Rightarrow \lambda_1 = 0,\ \lambda_2 = 2 + 3i.$$

where as in the previous problem we can pick only one of the two complext roots. The eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$$

$$v_2 = \begin{bmatrix} 1+i \\ -2 \\ 2 \end{bmatrix}.$$

The solution corresponding to $\lambda_1 = 0$ then becomes,

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$$

while the two solutions obtained from $\lambda_2 = 2 + 3i$ are

$$x_2 = \begin{bmatrix} \cos(3t) - \sin(3t) \\ -2\cos(3t) \\ 2\cos(3t) \end{bmatrix} e^{2t},$$

$$x_3 = \begin{bmatrix} \cos(3t) + \sin(3t) \\ -2\sin(3t) \\ 2\sin(3t) \end{bmatrix} e^{2t}.$$