VANDERBILT UNIVERSITY

MATH 2610 – ORDINARY DIFFERENTIAL EQUATIONS

Examples of sections 9.4 and 9.5

Question 1. Find the eigenvalues and eigenvectors of the following matrices:
(a)

$$A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}.$$

(b)

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Solutions.
a. Start with the characteristic equation

$$\det \begin{bmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{bmatrix} = -(3-\lambda)(2+\lambda)+4 = 0,$$

whose solutions are the eigenvalues

$$\lambda_1 = 2,\ \lambda_2 = -1.$$

Let us find the corresponding eigenvectors.
$\underline{\lambda_1 = 2}$:

$$\begin{bmatrix} 3-\lambda_1 & -2 \\ 2 & -2-\lambda_1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix},$$

hence we want to solve

$$\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} v_1 = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We find

$$v_1 = a\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

As we saw in class, we can drop the free variable a and write

$$v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

$\underline{\lambda_2 = -1}$:

$$\begin{bmatrix} 3-\lambda_2 & -2 \\ 2 & -2-\lambda_2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix},$$

hence we want to solve

$$\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} v_2 = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We find

$$v_2 = a \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Again, we drop the free variable a, obtaining

$$v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Summarizing, we have the following eigenvalues and eigenvectors:

$$\lambda_1 = 2,\ v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ \lambda_2 = -1,\ v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

b. Start with the characteristic equation

$$\det \begin{bmatrix} 1-\lambda & 1 & 2 \\ 1 & 2-\lambda & 1 \\ 2 & 1 & 1-\lambda \end{bmatrix} = (1-\lambda)\Big((2-\lambda)(1-\lambda)-1\Big) - (1-\lambda-2) + 2\Big(1-2(2-\lambda)\Big) = 0.$$

Rearranging,

$$\begin{aligned} (2-\lambda)(1-\lambda)^2 - 1 + \lambda + 1 + \lambda - 6 + 4\lambda &= (2-\lambda)(1-\lambda)^2 - 6(1-\lambda) \\ &= (1-\lambda)\Big((2-\lambda)(1-\lambda) - 6\Big) = 0. \end{aligned}$$

The eigenvalues are now easily found to be

$$\lambda_1 = 4,\ \lambda_2 = -1,\ \lambda_3 = 1.$$

Let us find the corresponding eigenvectors.
$\underline{\lambda_1 = 4}$:

$$\begin{bmatrix} 1-\lambda_1 & 1 & 2 \\ 1 & 2-\lambda_1 & 1 \\ 2 & 1 & 1-\lambda_1 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{bmatrix},$$

so we need to solve

$$\begin{bmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solving the system and ignoring the free variable as before we obtain

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Repeating the process for $\lambda_2 = -1$, $\lambda_3 = 1$ we find, respectively

$$v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},\ v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Summarizing, we have the following eigenvalues with corresponding eigenvectors

$$\lambda_1 = 4,\ v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ \lambda_2 = -1,\ v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},\ \lambda_3 = 1,\ v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$