

VANDERBILT UNIVERSITY

MATH 2610 – ORDINARY DIFFERENTIAL EQUATIONS

Examples of sections 4.6 and 4.7

**Question 1.** Knowing that

$$y_1 = x^2, \text{ and } y_2 = x^3$$

are two linearly independent solutions of

$$x^2 y'' - 4xy' + 6y = 0,$$

find the particular solution  $y_p$  of the equation

$$x^2 y'' - 4xy' + 6y = x^3. \tag{1}$$

**Question 2.** By directly plugging in the formula for  $y_p$ , namely,

$$y_p(x) = -y_1(x) \int \frac{y_2(x)f(x)}{W(x)} dx + y_2(x) \int \frac{y_1(x)f(x)}{W(x)} dx, \tag{2}$$

show that it yields a solution to

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x). \tag{3}$$

**Solutions.**

1. It is important to notice that (2) holds when the equation is written as in (3), i.e., with the coefficient of  $y''$  equal to 1. Thus, we divide (1) by  $x^2$ , obtaining

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = x,$$

in which case  $f(x) = x$ . We have

$$W = y_1 y_2' - y_2 y_1' = x^2 3x^2 - x^3 2x = x^4$$

(notice that  $W \neq 0$  since the equation is not defined for  $x = 0$ ). Then

$$\int \frac{y_2(x)f(x)}{W(x)} dx = \int \frac{x^3 x}{x^4} dx = x,$$

and

$$\int \frac{y_1(x)f(x)}{W(x)} dx = \int \frac{x^2 x}{x^4} dx = \ln x.$$

We thus find

$$y_p = x^3(\ln x - 1).$$

2. We have to plug (2) into (3). Since we shall differentiate  $y_p$ , it is useful to remember the Fundamental Theorem of Calculus, which gives

$$\left( \int \frac{y_2(x)f(x)}{W(x)} dx \right)' = \frac{y_2(x)f(x)}{W(x)},$$

and

$$\left( \int \frac{y_2(x)f(x)}{W(x)} dx \right)' = \frac{y_2(x)f(x)}{W(x)}.$$

Using these formulas and the product rule we find

$$y_p' = -y_1' \int \frac{y_2 f}{W} - y_1 \frac{y_2 f}{W} + y_2' \int \frac{y_1 f}{W} + y_2 \frac{y_1 f}{W},$$

where we write  $\int \frac{y_2 f}{W}$  instead of  $\int \frac{y_2(x)f(x)}{W(x)} dx$  in order to simplify the notation (analogously for the other integral). Taking another derivative

$$y_p'' = -y_1'' \int \frac{y_2 f}{W} - 2y_1' \frac{y_2 f}{W} - y_1 \left( \frac{y_2 f}{W} \right)' + y_2'' \int \frac{y_1 f}{W} + 2y_2' \frac{y_1 f}{W} + y_2 \left( \frac{y_1 f}{W} \right)'.$$

Using  $y_p$ ,  $y_p'$  and  $y_p''$  into the equation we find

$$\begin{aligned} y_p'' + py_p' + qy_p &= -(y_1'' + py_1' + qy_1) \int \frac{y_2 f}{W} + (y_2'' + py_2' + qy_2) \int \frac{y_2 f}{W} \\ &\quad - \frac{y_2 f}{W} (py_1 + 2y_1') + \frac{y_1 f}{W} (py_2 + 2y_2') - y_1 \left( \frac{y_2 f}{W} \right)' + y_2 \left( \frac{y_1 f}{W} \right)'. \end{aligned}$$

Since by hypothesis  $y_1$  and  $y_2$  are solutions of the homogeneous equation,

$$y_1'' + py_1' + qy_1 = 0,$$

and

$$y_2'' + py_2' + qy_2 = 0,$$

so

$$y_p'' + py_p' + qy_p = -\frac{y_2 f}{W} (py_1 + 2y_1') + \frac{y_1 f}{W} (py_2 + 2y_2') - y_1 \left( \frac{y_2 f}{W} \right)' + y_2 \left( \frac{y_1 f}{W} \right)'$$

By the product rule

$$\left( \frac{y_2 f}{W} \right)' = y_2' \frac{f}{W} + y_2 f' \frac{1}{W} + y_2 f \left( \frac{1}{W} \right)',$$

and

$$\left( \frac{y_1 f}{W} \right)' = y_1' \frac{f}{W} + y_1 f' \frac{1}{W} + y_1 f \left( \frac{1}{W} \right)'.$$

Therefore

$$\begin{aligned} y_p'' + py_p' + qy_p &= -\frac{y_2 f}{W} (py_1 + 2y_1') + \frac{y_1 f}{W} (py_2 + 2y_2') \\ &\quad - y_1 y_2' \frac{f}{W} - y_1 y_2 f' \frac{1}{W} - y_1 y_2 f \left( \frac{1}{W} \right)' \\ &\quad + y_2 y_1' \frac{f}{W} + y_2 y_1 f' \frac{1}{W} + y_2 y_1 f \left( \frac{1}{W} \right)'. \end{aligned}$$

Notice that the last two terms of the second line cancel with the last two terms of the third line. We are left with

$$\begin{aligned} y_p'' + py_p' + qy_p &= -\frac{y_2 f}{W} (py_1 + 2y_1') + \frac{y_1 f}{W} (py_2 + 2y_2') - y_1 y_2' \frac{f}{W} + y_2 y_1' \frac{f}{W} \\ &= -p \frac{y_2 f}{W} y_1 - 2 \frac{y_2 f}{W} y_1' + p \frac{y_1 f}{W} y_2 + 2 \frac{y_1 f}{W} y_2' - y_1 y_2' \frac{f}{W} + y_2 y_1' \frac{f}{W}. \end{aligned}$$

The first and third terms on the last line cancel out, and then

$$\begin{aligned}y_p'' + py_p' + qy_p &= -2\frac{y_2f}{W}y_1' + 2\frac{y_1f}{W}y_2' - y_1y_2'\frac{f}{W} + y_2y_1'\frac{f}{W} \\ &= -2\frac{f}{W}y_2y_1' + 2\frac{f}{W}y_1y_2' - y_1y_2'\frac{f}{W} + y_2y_1'\frac{f}{W} \\ &= \frac{f}{W}y_1y_2' - \frac{f}{W}y_2y_1' = \frac{f}{W}(y_1y_2' - y_2y_1') \\ &= f,\end{aligned}$$

where in the last step we used that  $W = y_1y_2' - y_2y_1'$ .